Bijective rigid motions of the 2D Cartesian grid

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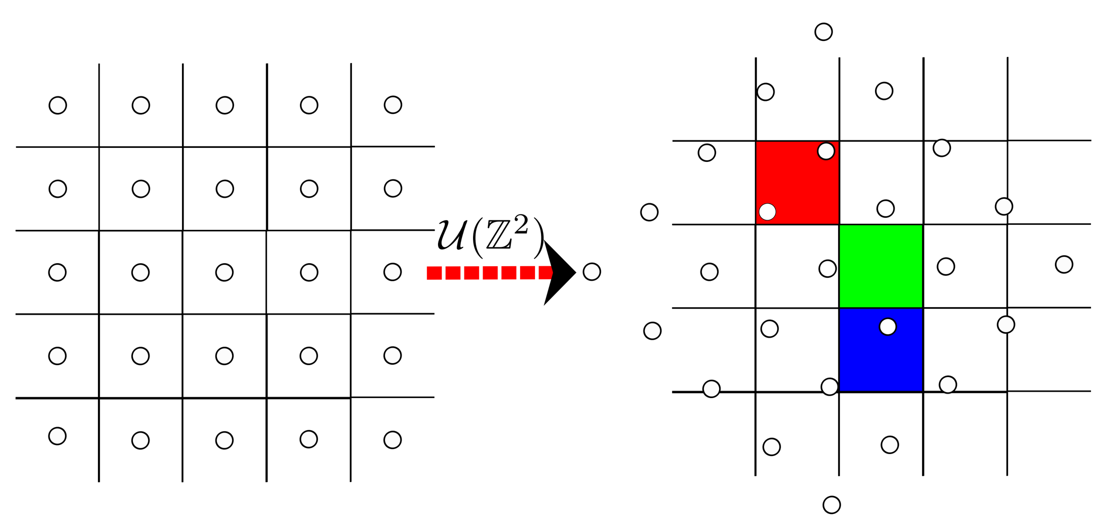
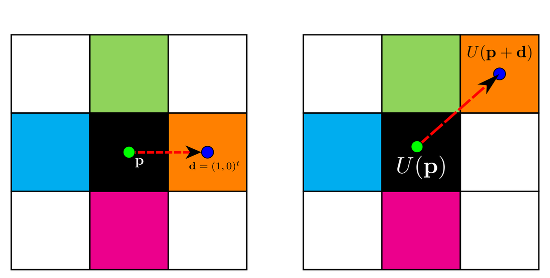
The Big Question

A digitized rigid motion U is defined as the composition of continuous rigid motion $\mathcal{U}$ followed by rounding function. They are neither injective nor surjective, in general. **What are the bijective digitized rigid motions?**



Approach

We observe local alterations of $\mathbb{Z}^2$ while using so-called *neighbourhood motion maps* which allow us to track changes between neighbourhoods of $\mathbf{p}$ and $U(\mathbf{p})$.



Neighbourhood motion maps evolve under U while the position of $\mathcal{U}(\mathbf{p})$ inside a digitization cell changes. Therefore, we consider the mapping by so-called *remainder map* $\rho(\mathbf{p}) = \mathcal{U}(\mathbf{p}) - U(\mathbf{p})$ which maps into the so-called *remainder range*.

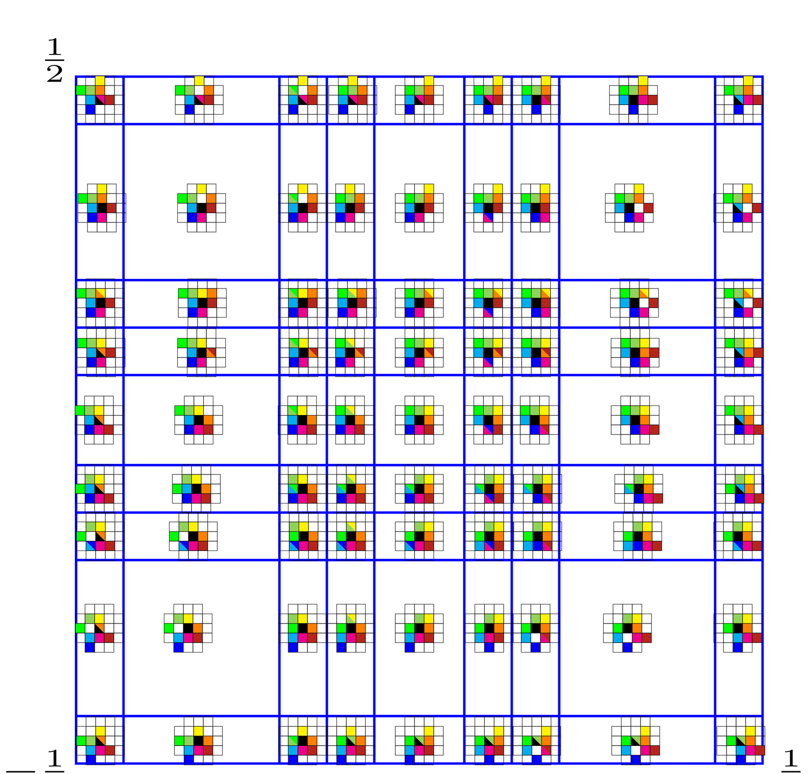


Figure: While $\rho(\mathbf{p})$ stays in the same zone, corresponding neighbourhood motion map does not change.

Bijective digitized rigid motions

To answer "The Big Question" we consider zones in the remainder range where we observe lack of surjectivity in the corresponding neighbourhood motion map.

The Answer

Bijective digitized rigid motions are those which are compositions of bijective digitized rotations—which are defined by Pythagorean twin triples—followed by translations $\mathbf{t} = \mathbf{t}' + \mathbb{Z}\boldsymbol{\psi} + \mathbb{Z}\boldsymbol{\omega}$, where $\mathbf{t}' \in \left(-\frac{1}{2(p^2+q^2)}, \frac{1}{2(p^2+q^2)}\right)^2$.

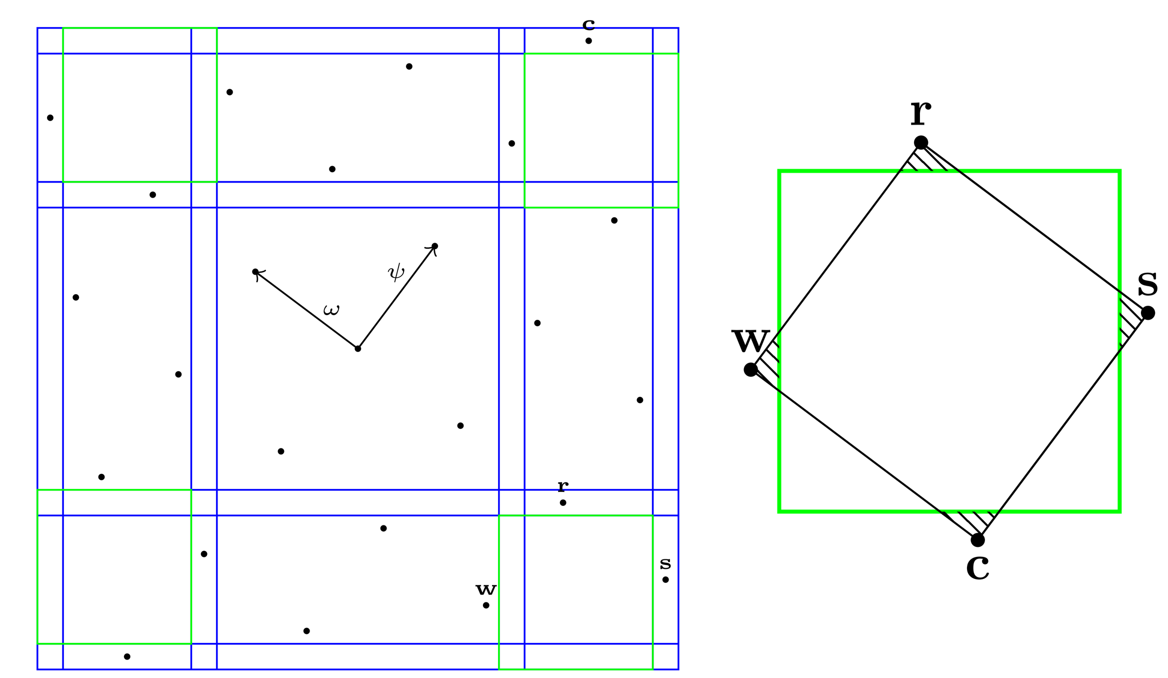


Figure: On the left, the mapping $\rho(\mathbb{Z}^2)$. When a digitized rigid motion is given by a primitive Pythagorean triple $(p^2-q^2, 2pq, p^2+q^2)$, these points belong to a finite cyclic group. On the right, in green, zoomed non-surjective frame with the black-hatched region $\left(-\frac{1}{2(p^2+q^2)}, \frac{1}{2(p^2+q^2)}\right)^2$.

Bijective digitized motions of finite sets

Bijective digitized rigid motions are not dense in general. For example, there is no other bijective digitized rotations between angles 22.62° and 36.87°. This motivates us to propose algorithms for verifying if an application of a digitized rigid motion to a finite set S is bijective. In this approach we consider non-injective zones in the remainder range.

Forward algorithm

This approach consists of verifying if there is $\mathbf{p} \in S \subset \mathbb{Z}^2$, such that $\rho(\mathbf{p})$ is in the non-injective zones.

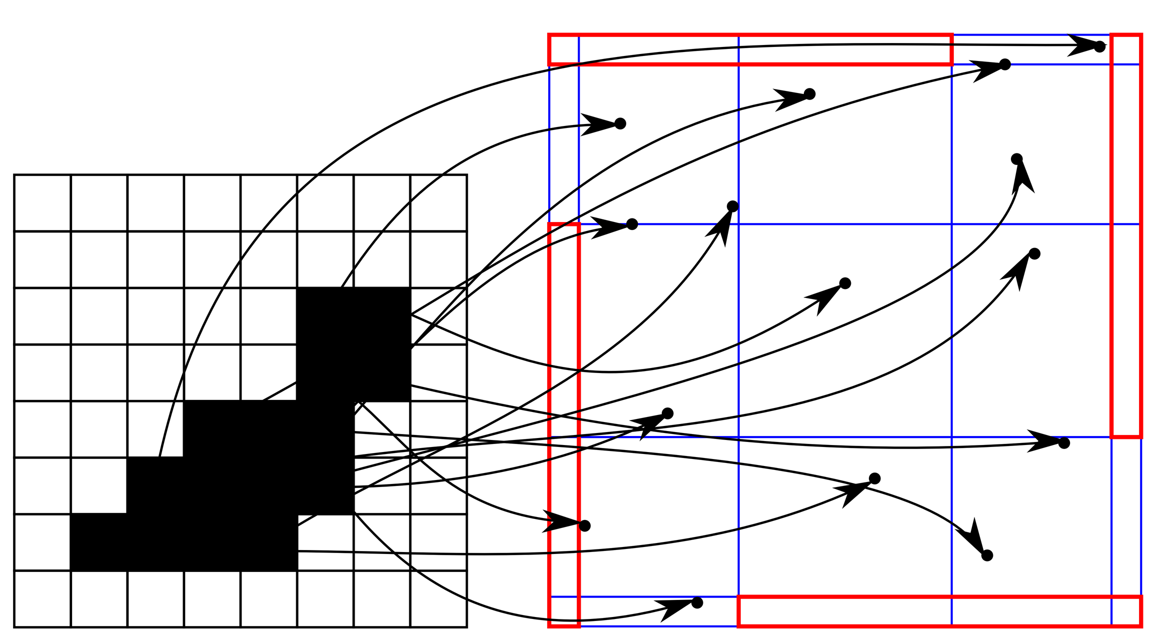


Figure: Graphical interpretation of forward algorithm. Each arrow represents a mapping via $\rho(\mathbf{p})$.

Backward algorithm

In this approach, for each point inside the non-injective zones, we find a lattice of its preimages. Then we find an intersection of this lattice with a finite set S.

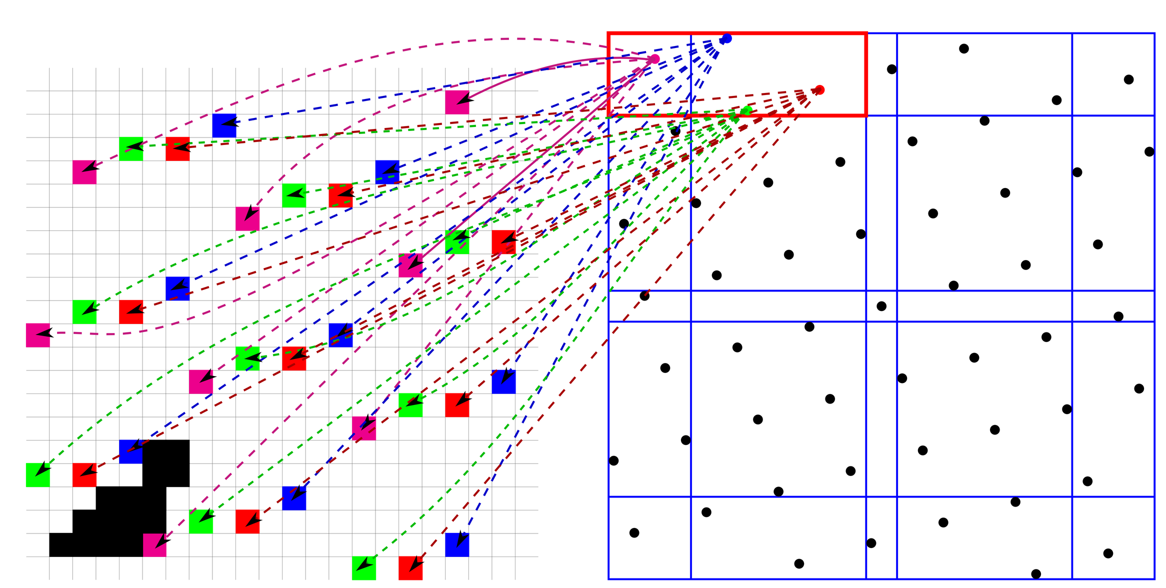
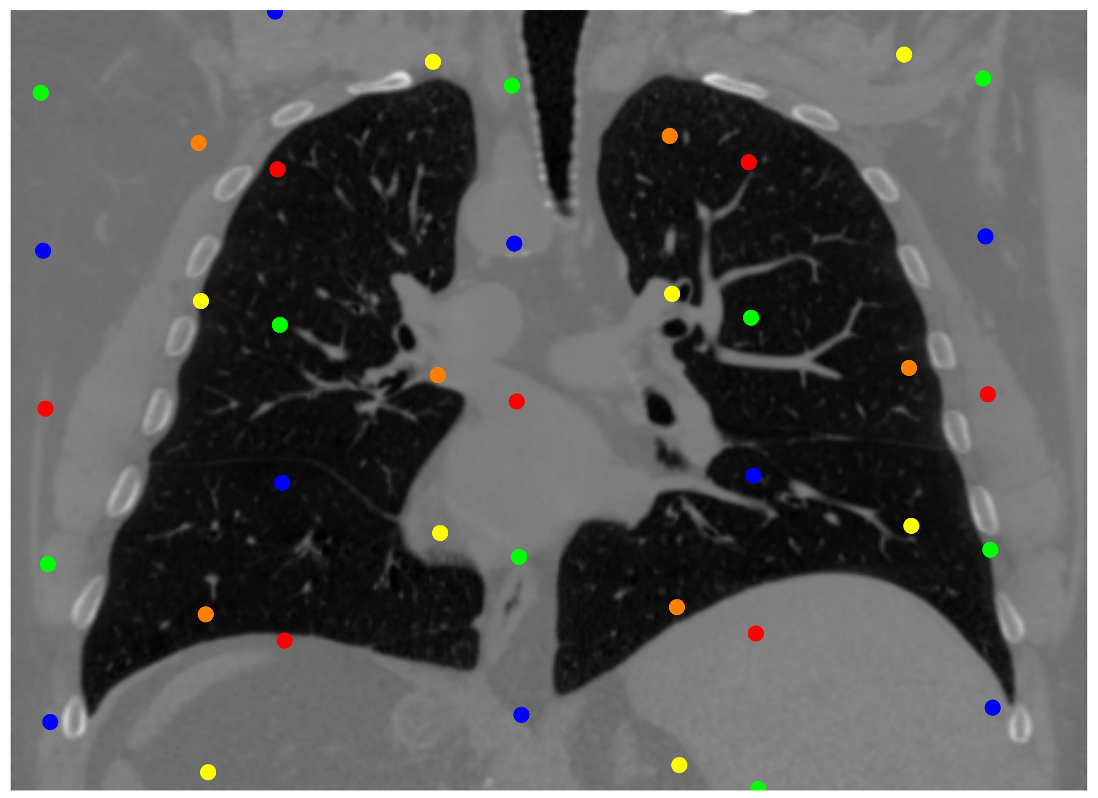


Figure: Graphical interpretation of backward algorithm. Each arrow represents a mapping from a point in a non-injective zone into a point of a lattice of its preimages.

Application

One possible application of backward algorithm is to check if a digitized rigid motion alters important parts of an image.



Conclusions

We proved some necessary and sufficient conditions of bijective rigid motions on $\mathbb{Z}^2$. From a more practical point of view, we proposed two efficient algorithms for verifying whether a given digitized rigid motion is bijective when restricted to a finite set. The complexities of the forward and backward algorithms are $O(|S|)$ and $O(q) + O(\log \min(p^2 - q^2, 2pq)) + O(\sqrt{|S|})$, respectively.

References

[1] Nouvel, B. and Rémila, E., *Characterization of bijective discretized rotations*, Lecture Notes in Computer Science (2004).

[2] Nouvel, B. and Rémila, E., *Configurations induced by discrete rotations: Periodicity and quasi-periodicity properties*, Discrete Applied Mathematics (2005).

Acknowledgements

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