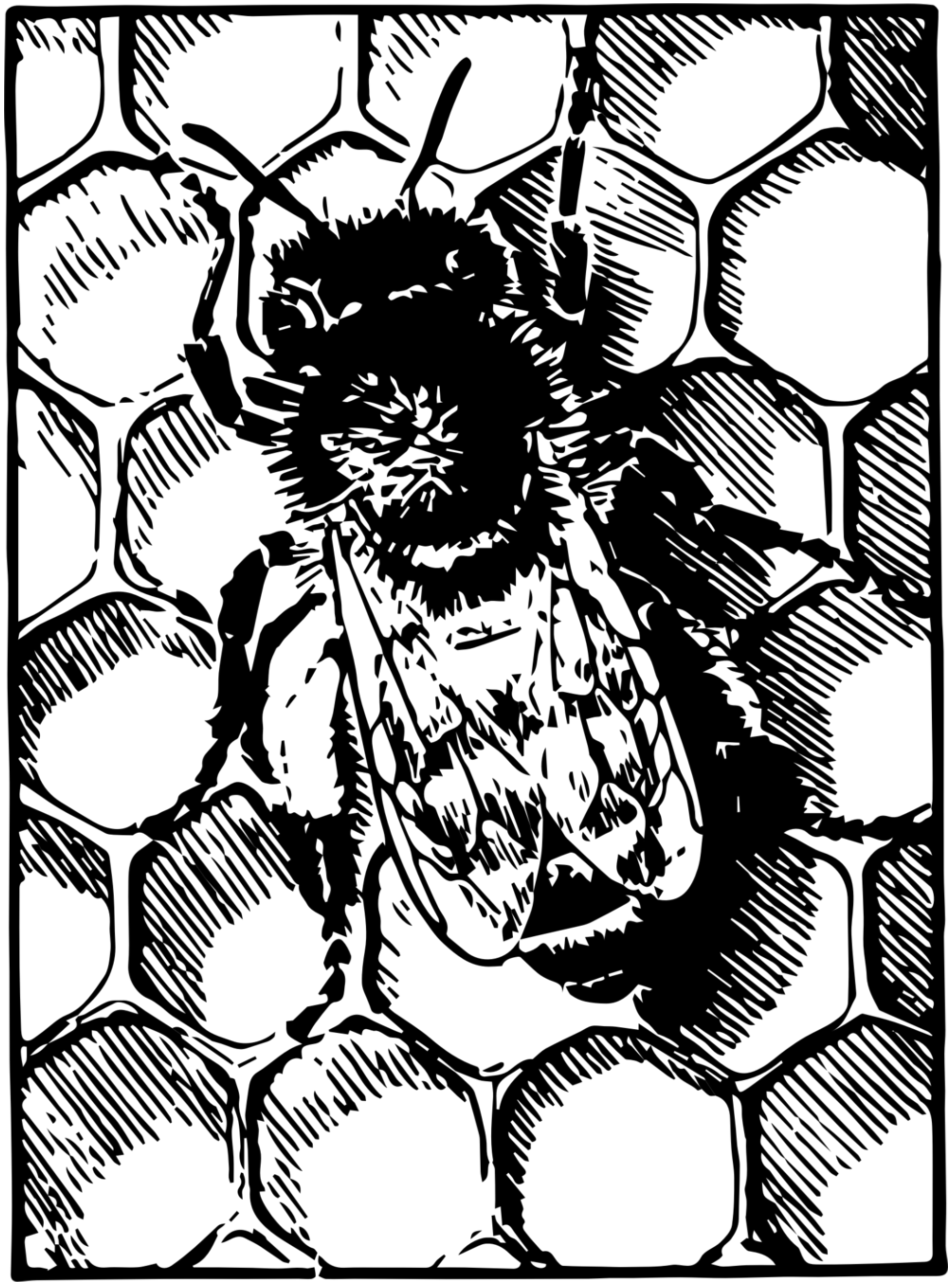
Honeycomb Geometry

Rigid Motions on the Hexagonal Grid

by Kacper Pluta, Pascal Romon,

Yukiko Kenmochi and Nicolas Passat



Motivations

We came to agree with Nouvel & Rémila that digitized rigid motions defined on the square grid are burdened with a fundamental incompatibility between rotations and the geometry of the grid.

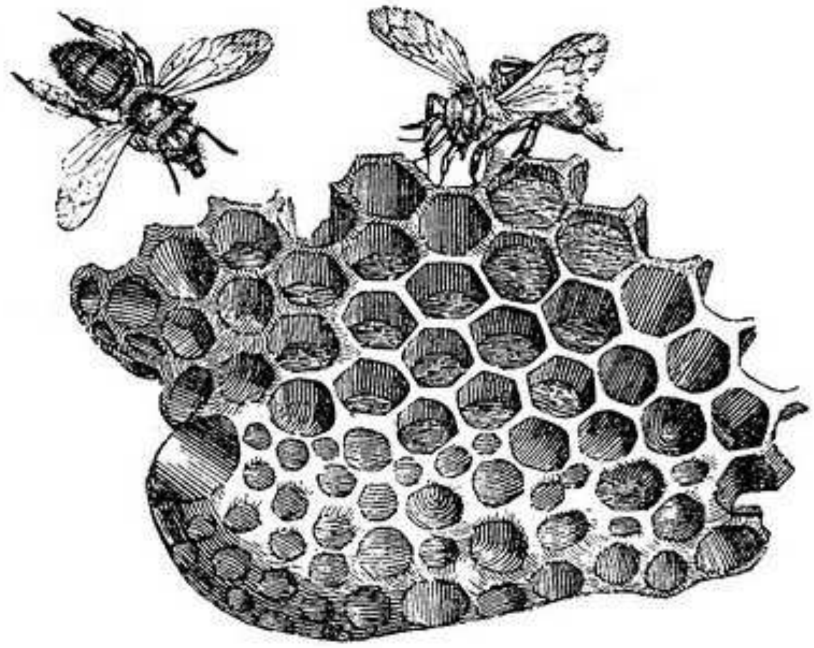
Agenda

- Introduction to the Bees' Point of View
- Quick Introduction to Rigid Motions
- Neighborhood Motion Maps
- Contributions
- Conclusions & Perspectives

The beehive figure's source and author unknown (if you recognize it, please let me know). The image of the bee comes from http://karenswhimsy.com/public-domain-images (public domain)

Introduction to the Bees' Point of View

Or why bees are right



The figure comes from http://thegraphicsfairy.com/vintage-clip-art-bees-with-honeycomb

Pros and Cons

Square grid

+ Memory addressing
+ Sampling is easy to define
- Sampling is not optimal (ask bees)
- Neighbors are not equidistant
- Connectivity paradox

Hexagonal grid

+ Uniform connectivity
+ Equidistant neighbors
+ Sampling is optimal
- Memory addressing is not trivial
- Sampling is difficult to define

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Howdy vision lads and gals! These problems seem to be somehow solved.

The figure by Pearson Scott Foresman, Wikimedia.

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Hexagonal grid

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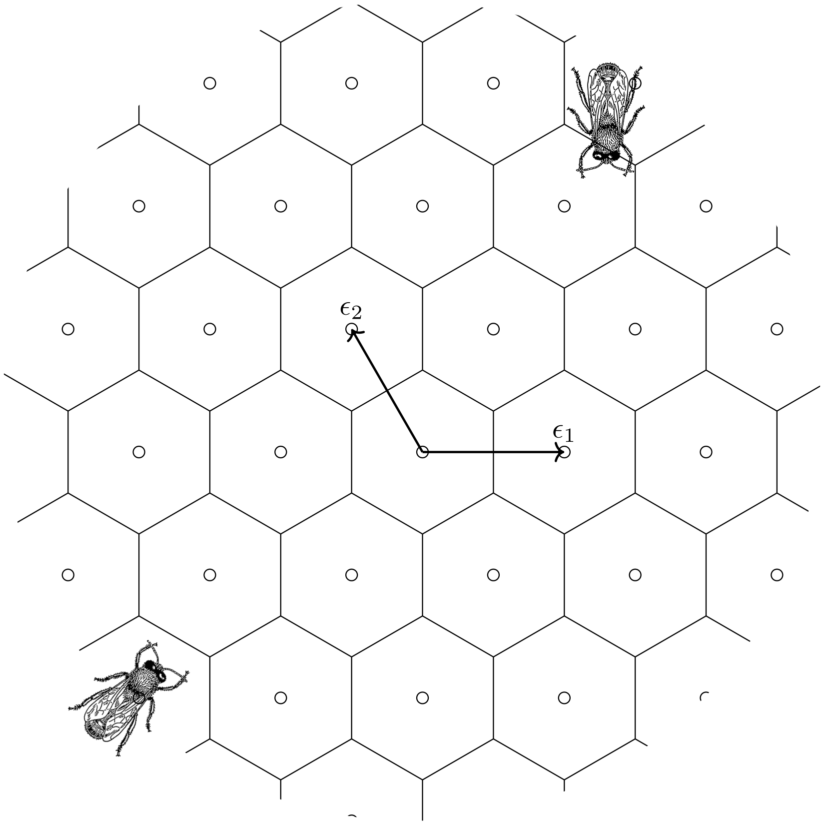
~ Sampling is difficult to define

You may think: "Hold your horses! It’s not a bug, it’s a feature..."

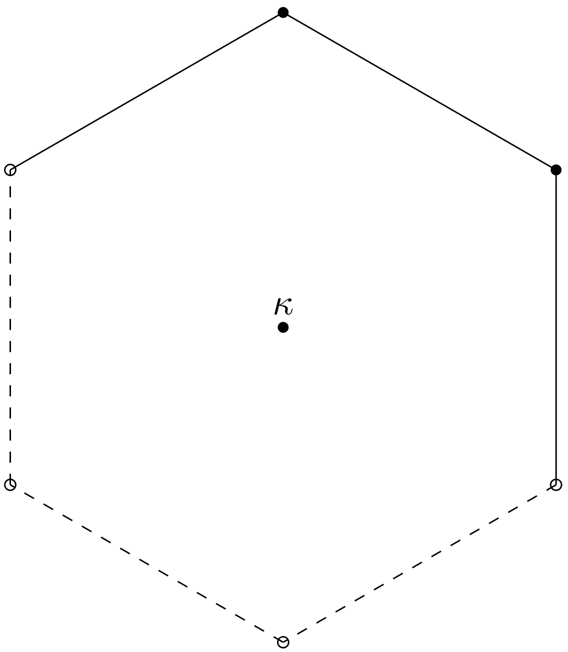
The figure by Pearson Scott Foresman, Wikimedia.

Hexagonal Grid



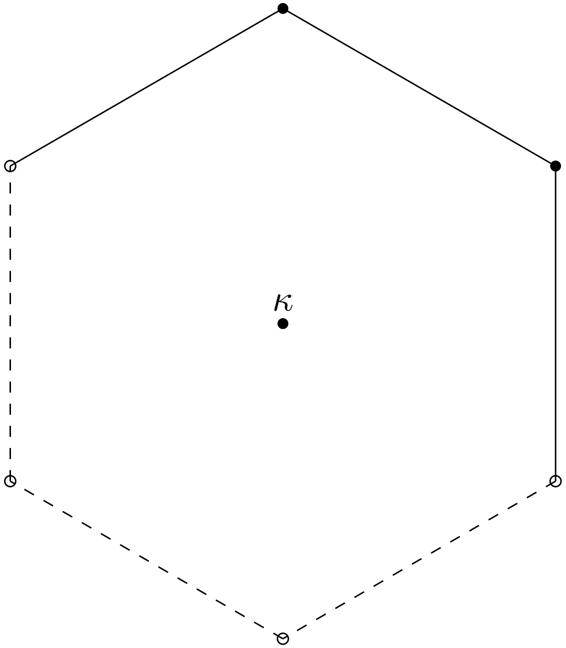
The hexagonal lattice: $\Lambda = \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2$ and the hexagonal grid $\mathcal{H}$

Digitization Model



The digitization operator is defined as a function $\mathcal{D} : \mathbb{R}^2 \to \Lambda$ such that $\forall \mathbf{x} \in \mathbb{R}^2, \exists ! \mathcal{D}(\mathbf{x}) \in \Lambda$ and $\mathbf{x} \in \mathcal{C}(\mathcal{D}(\mathbf{x}))$.

Digitization Model



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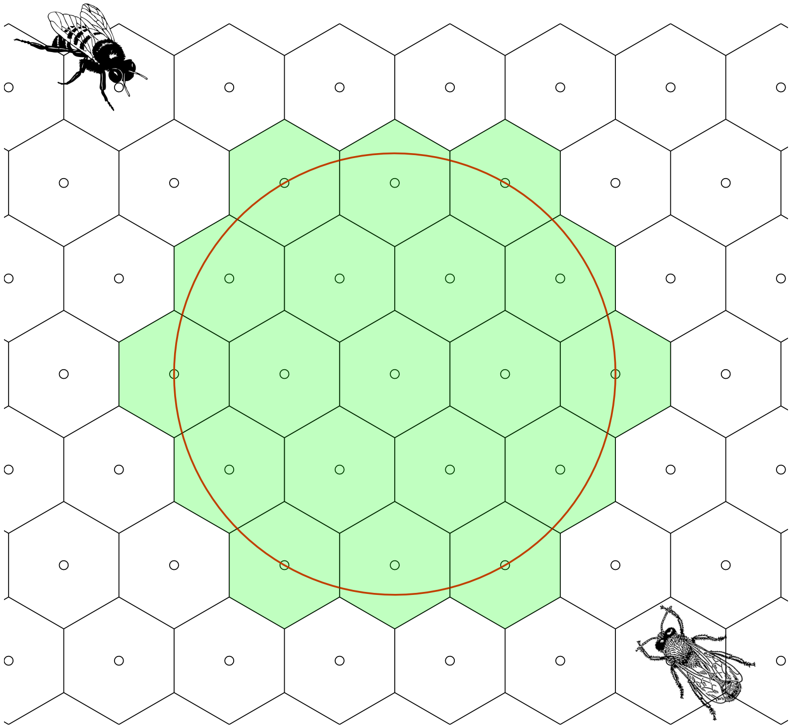
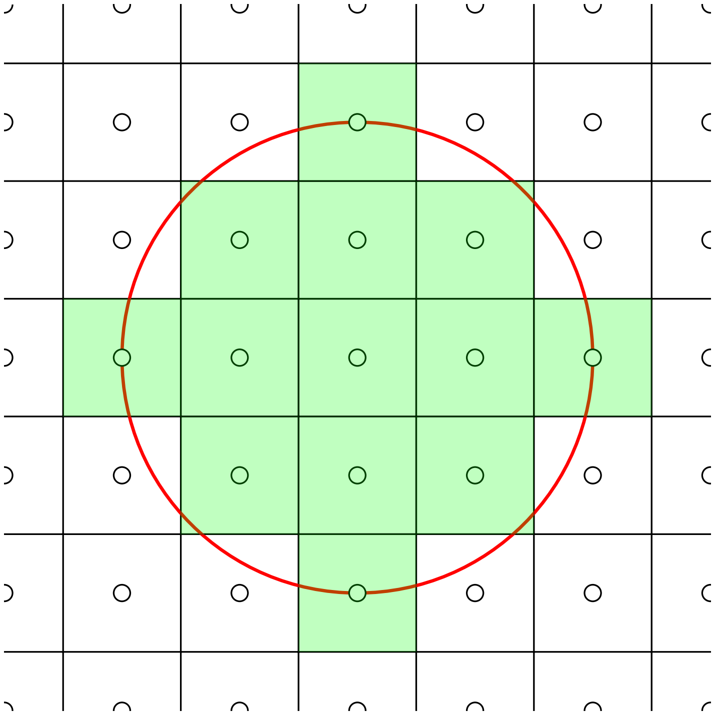
This is a definition for digital geometers not for computer vision guys...

How many digital balls do you see?

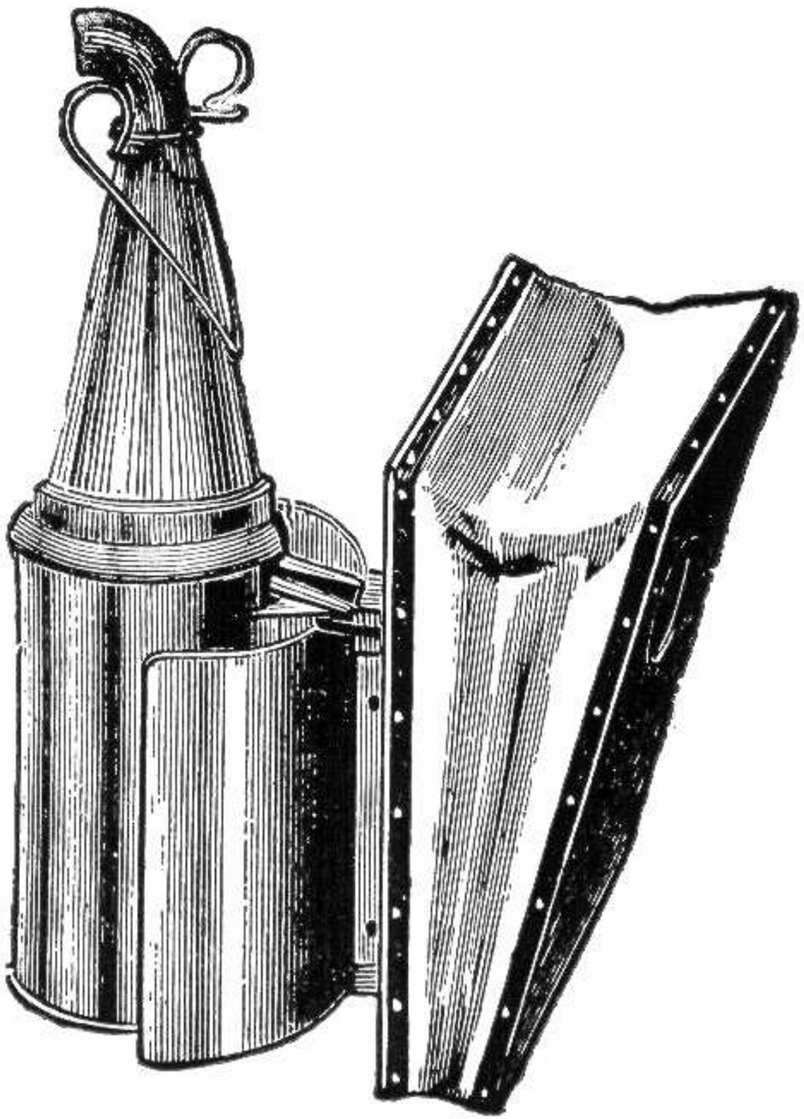




The figure of bumble bee comes from http://www.ase.org.uk (public domain)

Quick Lesson on Rigid Motions

Or how to become a beekeeper. Part I - Equipment



The figure comes from Wikimedia. Original source *The New Student's Reference Work* *(public domain)*

Rigid Motions on $\mathbb{R}^2$

$$\begin{array}{lllll} \mathcal{U} & : & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & & \mathbf{x} & \mapsto & \mathbf{R}\mathbf{x} + \mathbf{t} \end{array}$$

Properties

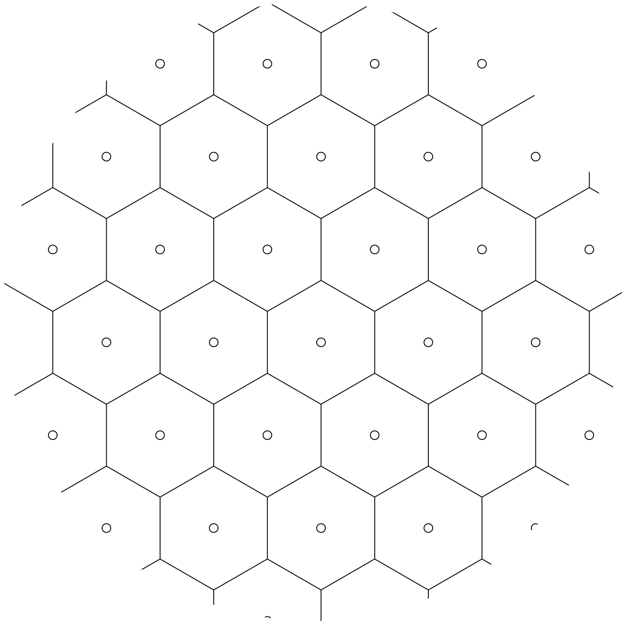
- Isometry map - distance preserving map
- Bijective

Rigid Motions on Λ

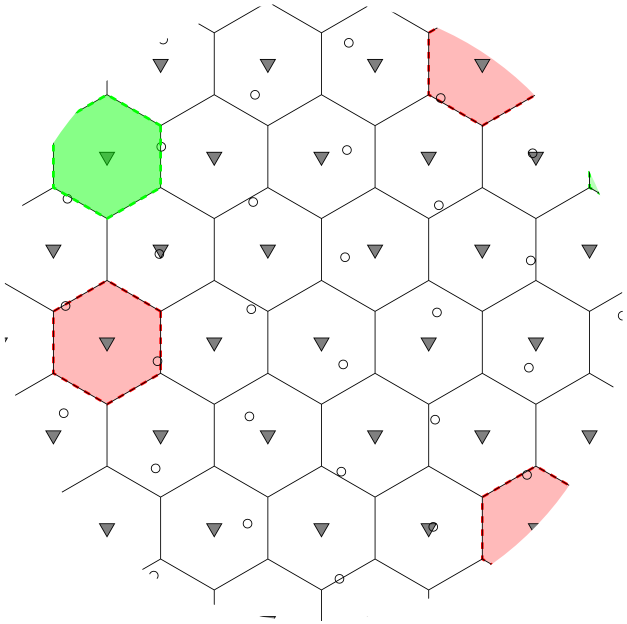
$$U = \mathcal{D} \circ \mathcal{U}_{|\Lambda}$$

Properties

- Do not preserve distances
- Non-injective
- Non-surjective



$\mathcal{U}(\Lambda)$



Related Studies

- Nouvel, B., Rémila, E.: On colorations induced by discrete rotations. In: DGCI, Proceedings. *Volume 2886 of Lecture Notes in Computer Science*., Springer (2003) 174–183

- Pluta, K., Romon, P., Kenmochi, Y., Passat, N.: Bijective digitized rigid motions on subsets of the plane. *Journal of Mathematical Imaging and Vision* (2017)

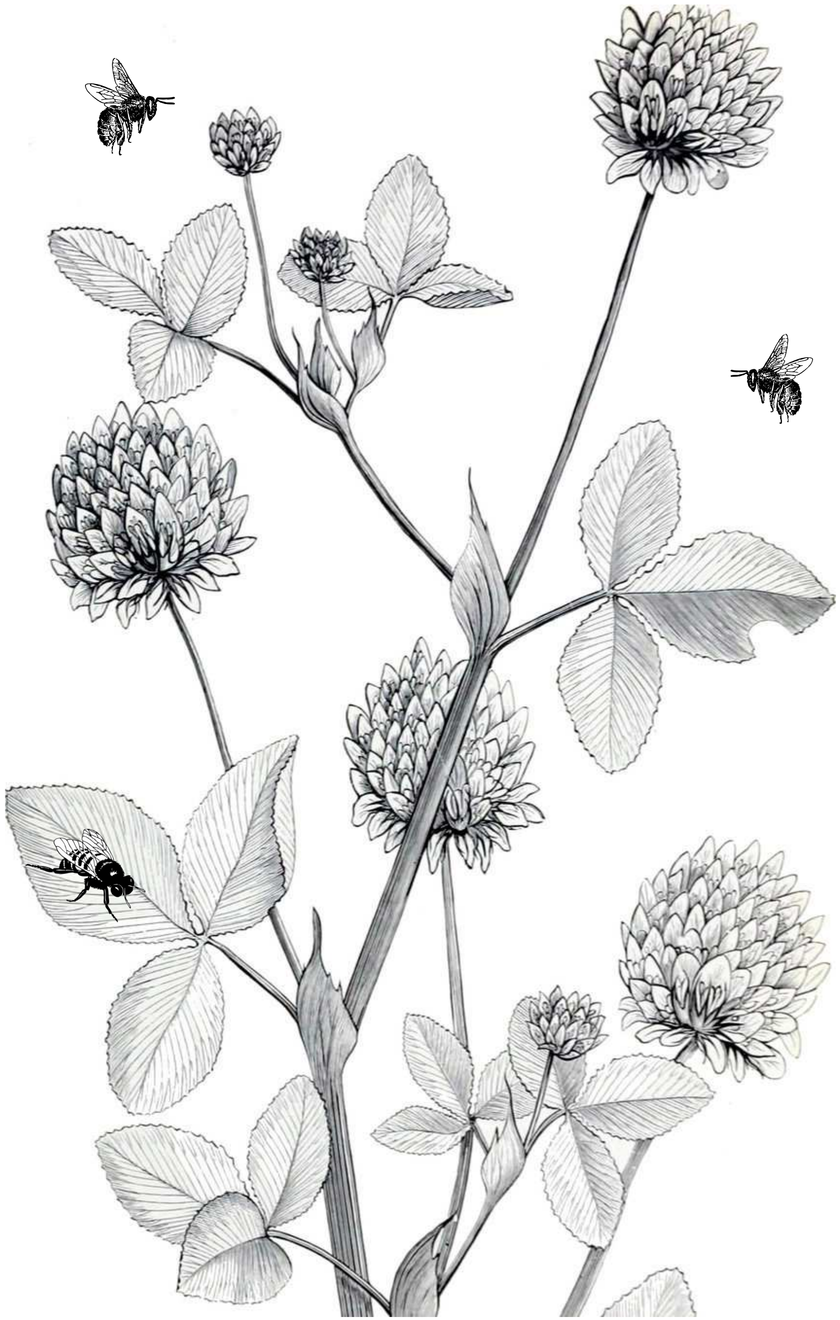
Contributions in Short

Pure extracted honey

- Extension of the former framework to the hexagonal grid
- Comparison of the loss of information between the hexagonal and square grids
- Complete set of neighborhood motion maps
- Source code of a tool to study digitized rigid motions on the hexagonal grid

Neighborhood Motion Maps

Or a manual of instructions in apiculture

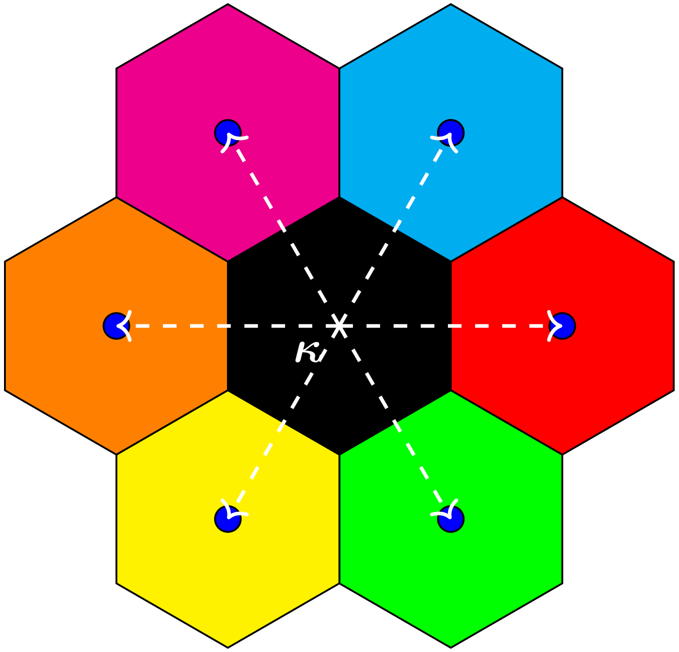


The figure comes from Wikimedia. The original source The honey bee: a manual of instruction in apiculture (public domain)

Neighborhood

The *neighborhood* of $\kappa \in \Lambda$ (of squared radius $r \in \mathbb{R}_+$)

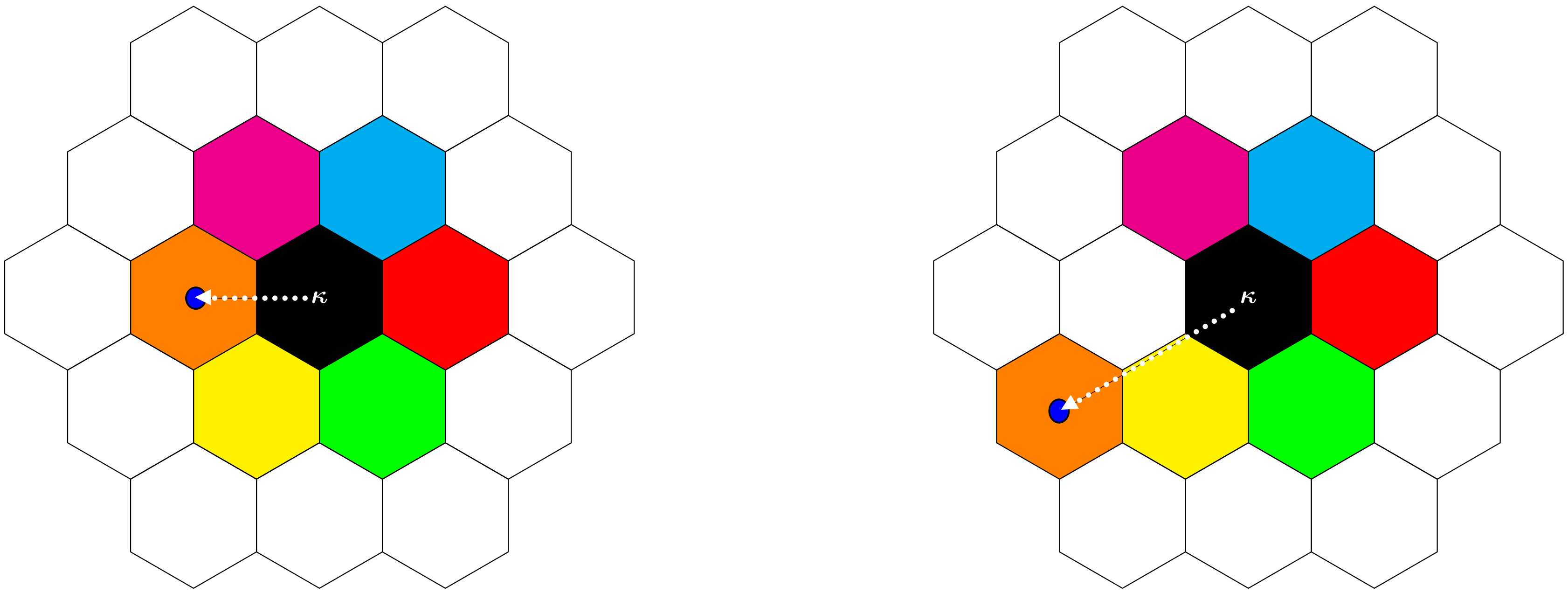
$$\mathcal{N}_r(\kappa) = \{\kappa + \delta \in \Lambda \mid \|\delta\|^2 \leq r\}$$



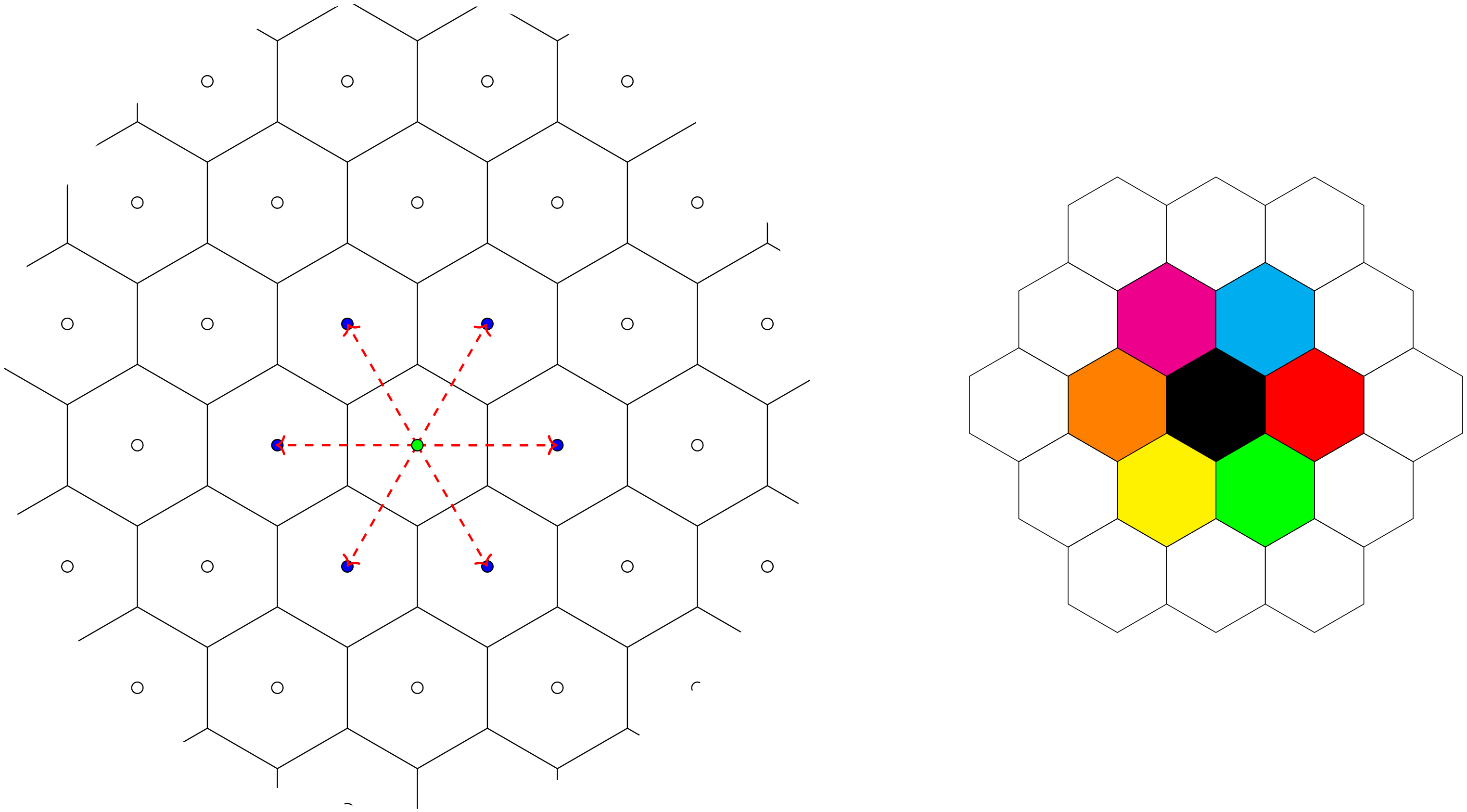
Neighborhood Motion Maps

The *neighborhood motion map* of $\kappa \in \Lambda$ with respect to U and $r \in \mathbb{R}_+$ is the function

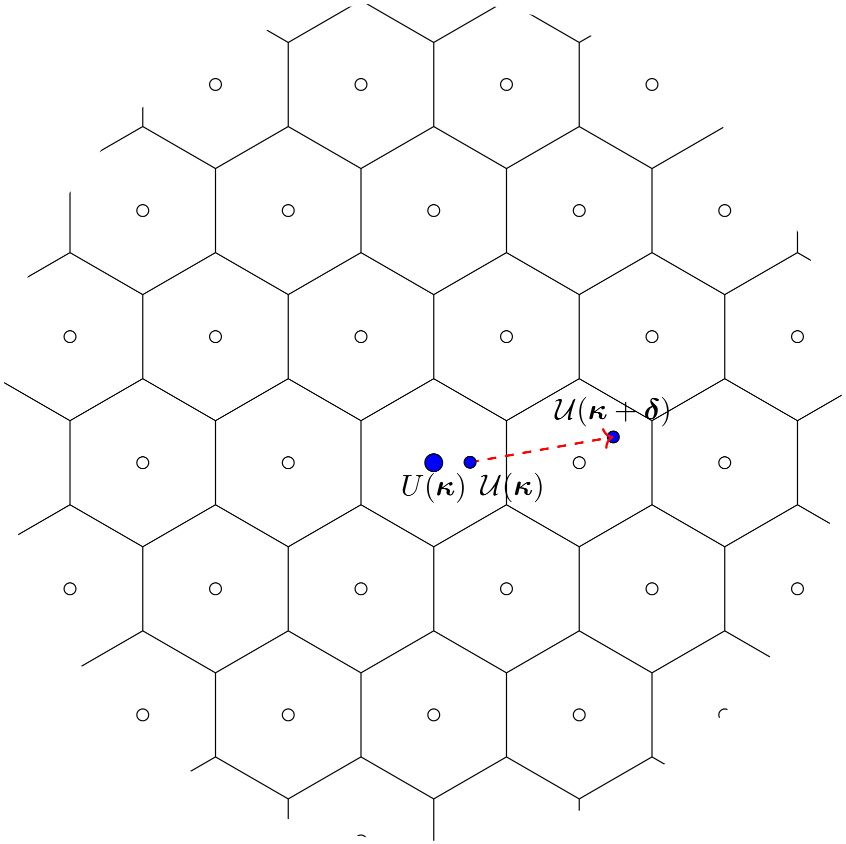
$$\left|\begin{array}{lcll} \mathcal{G}_r^U & : & \mathcal{N}_r(0) & \to & \mathcal{N}_{r'}(0) \\ & & \delta & \mapsto & U(\kappa + \delta) - U(\kappa). \end{array}\right.$$

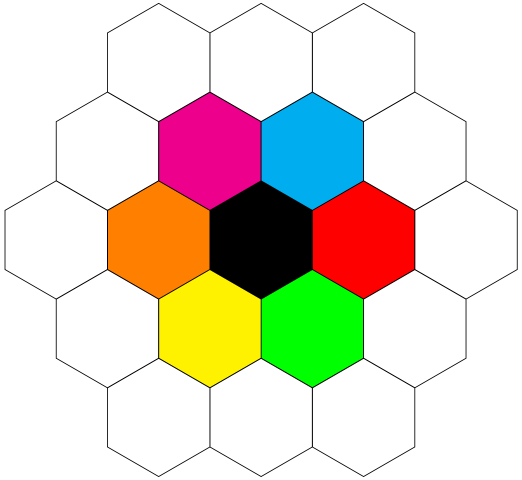


Remainder Map step-by-step



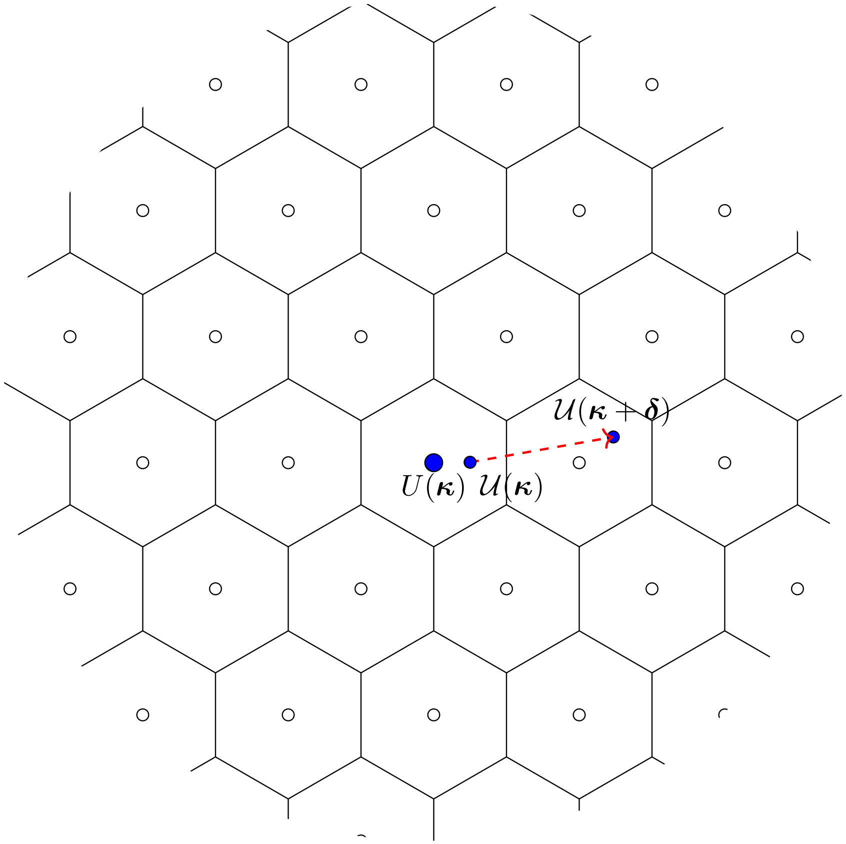
Remainder Map step-by-step

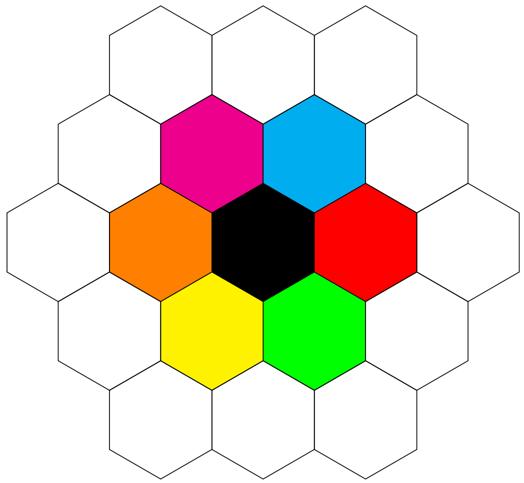




$$\mathcal{U}(\kappa + \delta) = \mathbf{R}\delta + \mathcal{U}(\kappa)$$

Remainder Map step-by-step

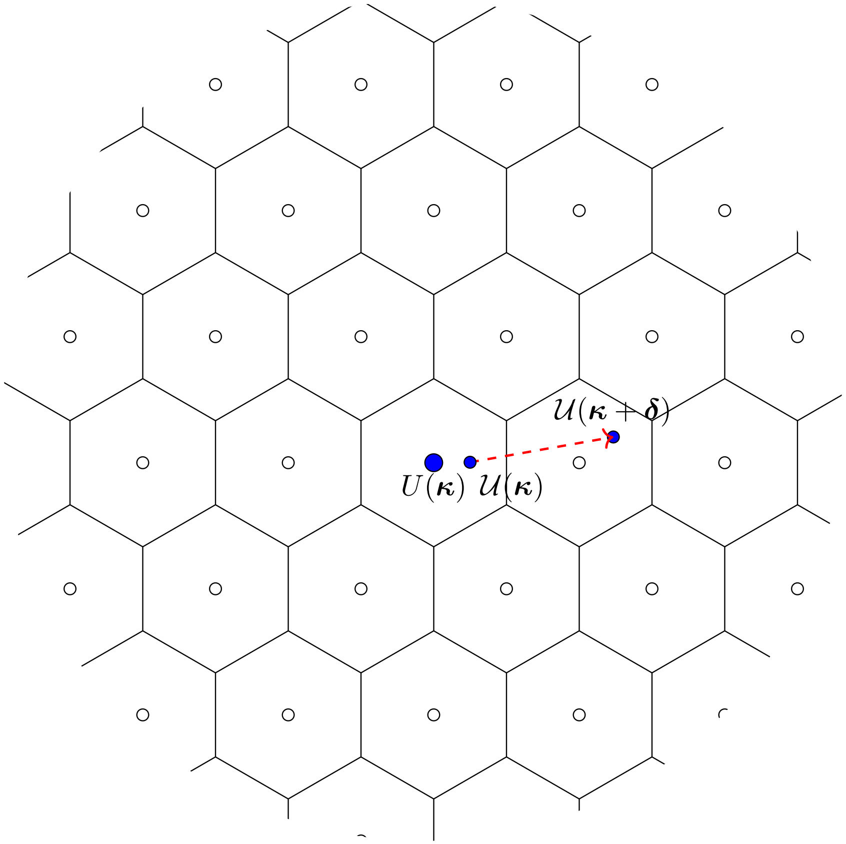


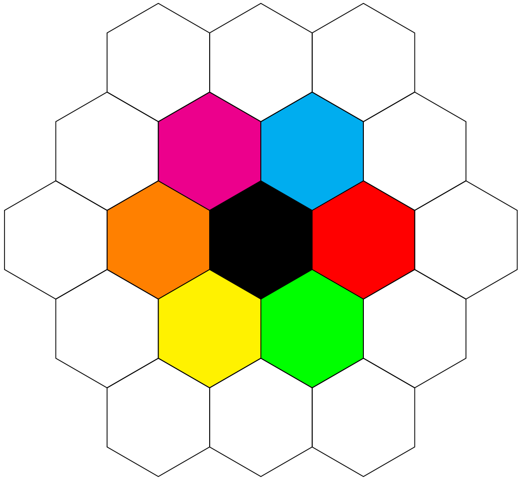


Without loss of generality, $U(\kappa)$ is an origin, then

$$\mathcal{U}(\delta) = \mathbf{R}\delta + \mathcal{U}(\kappa) - U(\kappa)$$

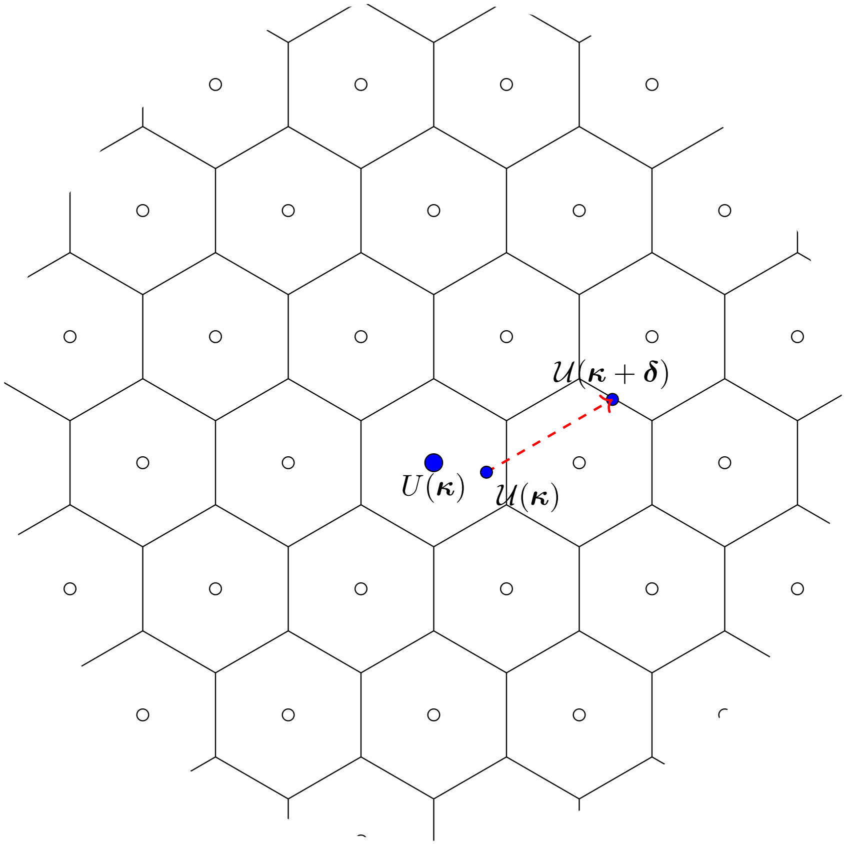
Remainder Map step-by-step

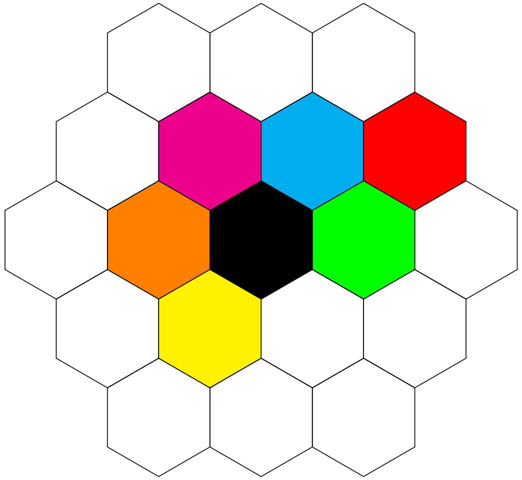




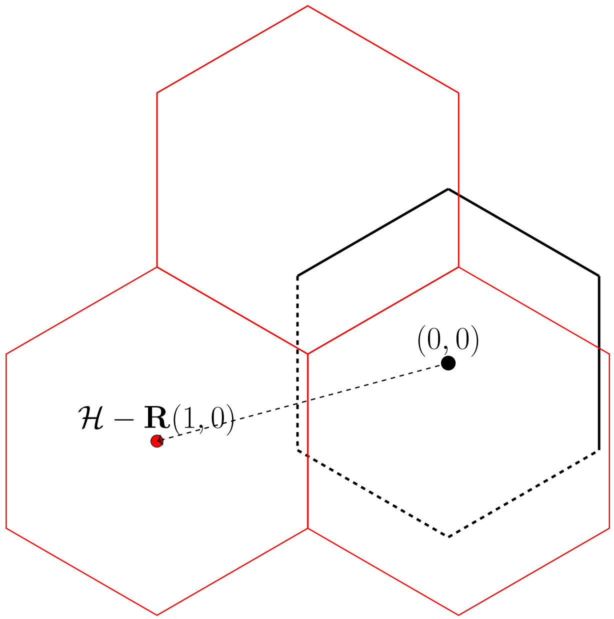
Remainder map defined as $\mathcal{F}(\boldsymbol{\kappa}) = \mathcal{U}(\boldsymbol{\kappa}) - U(\boldsymbol{\kappa}) \in \mathcal{C}(\mathbf{0})$ where the range $\mathcal{C}(\mathbf{0})$ is called the remainder range.

Remainder Map step-by-step



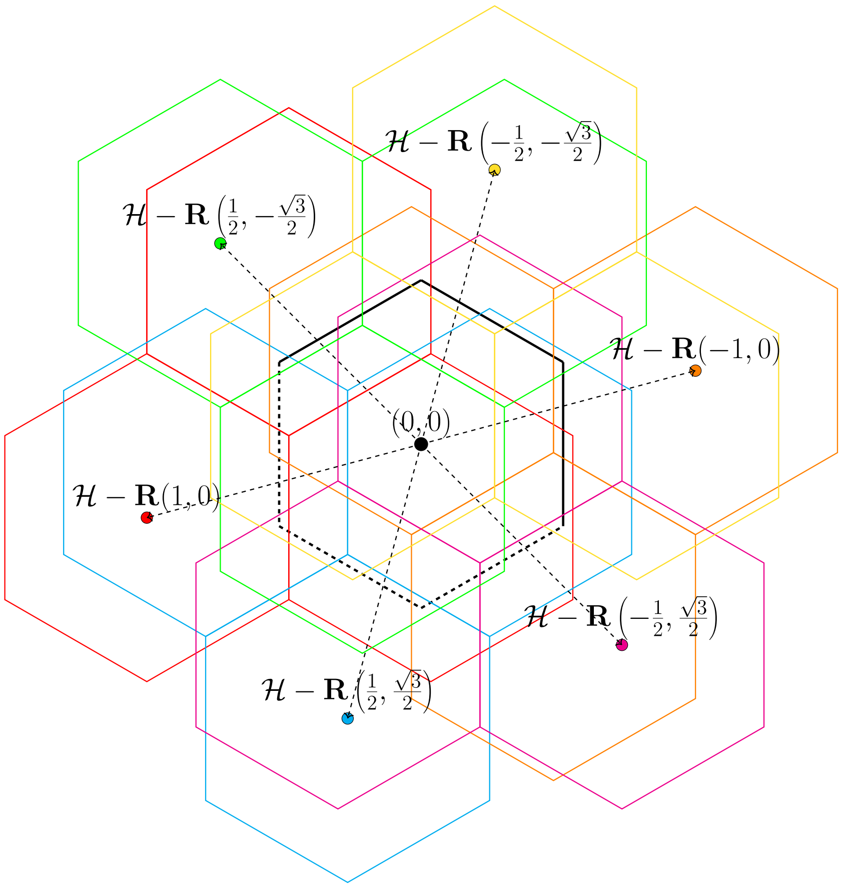


Remainder Map and Critical Rigid Motions



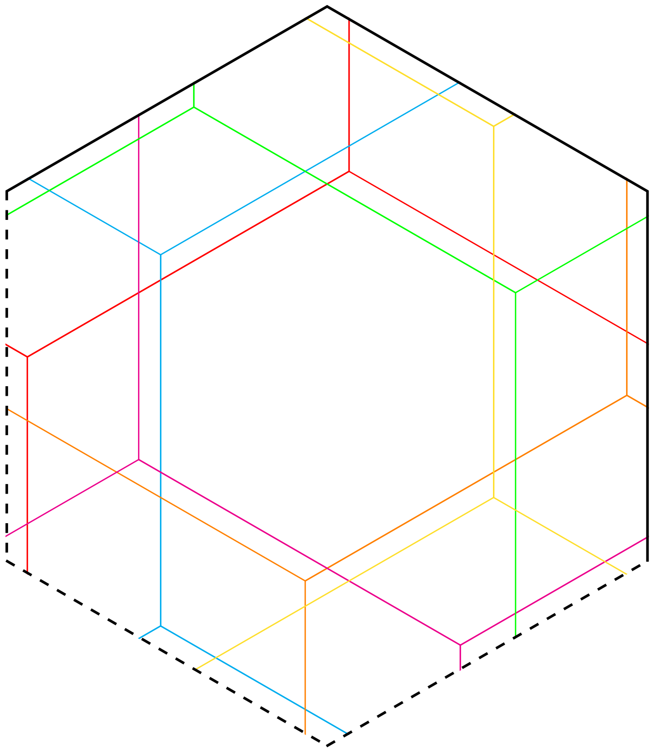
Critical cases can be observed via the relative positions of $\mathcal{F}(\kappa)$ which are formulated by the translation $\mathcal{H}-\mathbf{R}\delta$ that is to say $\mathcal{C}(\mathbf{0}) \cap (\mathcal{H}-\mathbf{R}\delta)$.

Remainder Map and Critical Rigid Motions



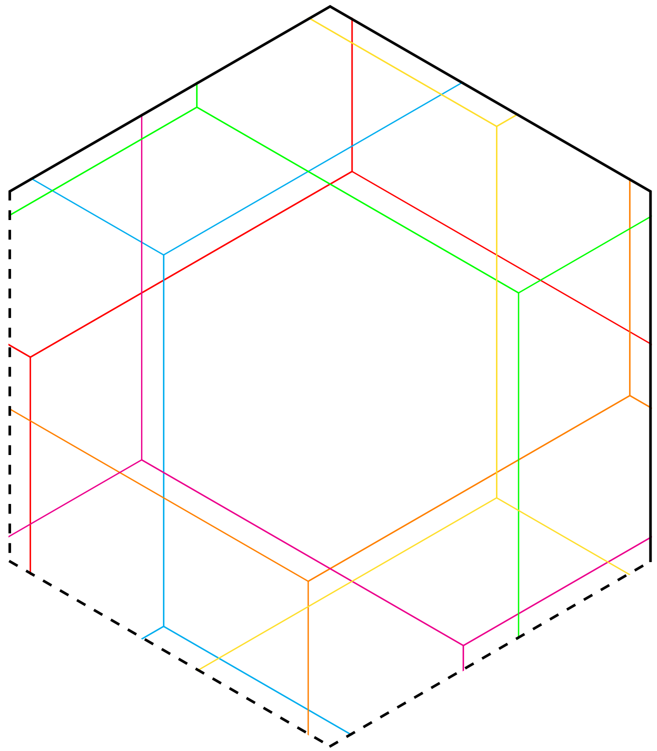
$$\mathscr{H} = \bigcup_{\delta \in \mathcal{N}_r(\mathbf{0})} (\mathcal{H} - \mathbf{R}\delta) \cap \mathcal{C}(\mathbf{0})$$

Frames



Each region bounded by critical lines is called a frame.

Frames

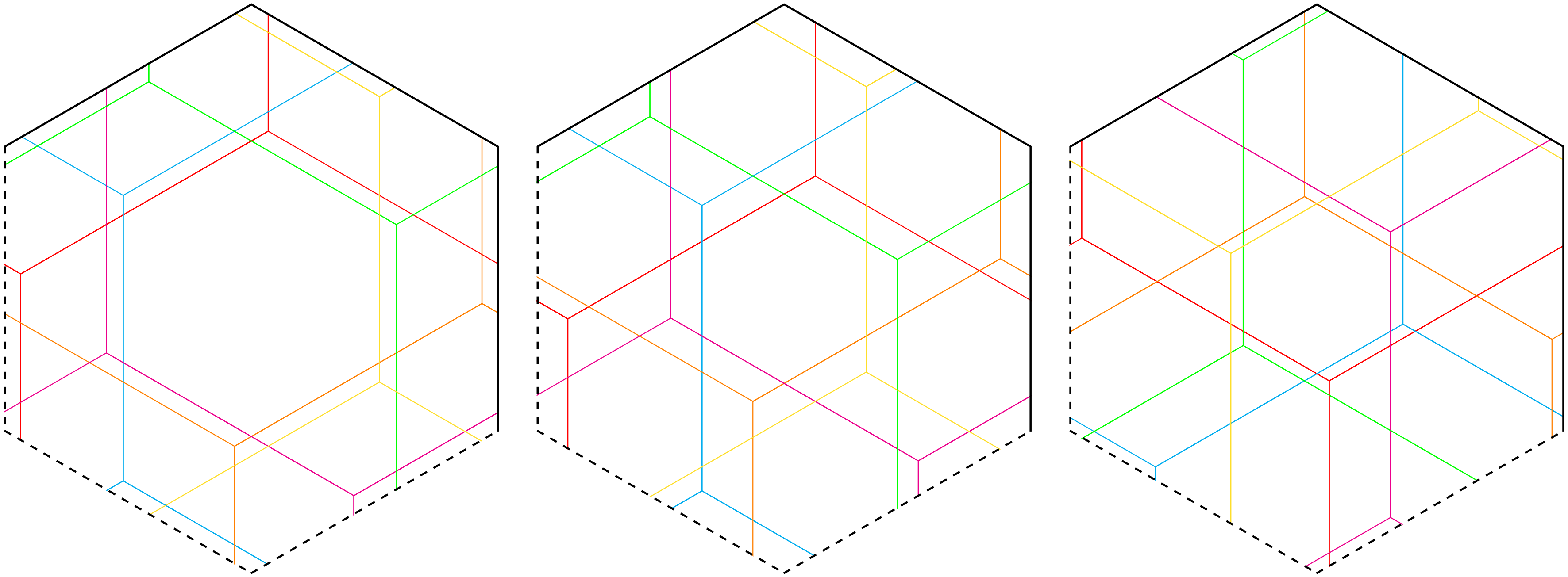


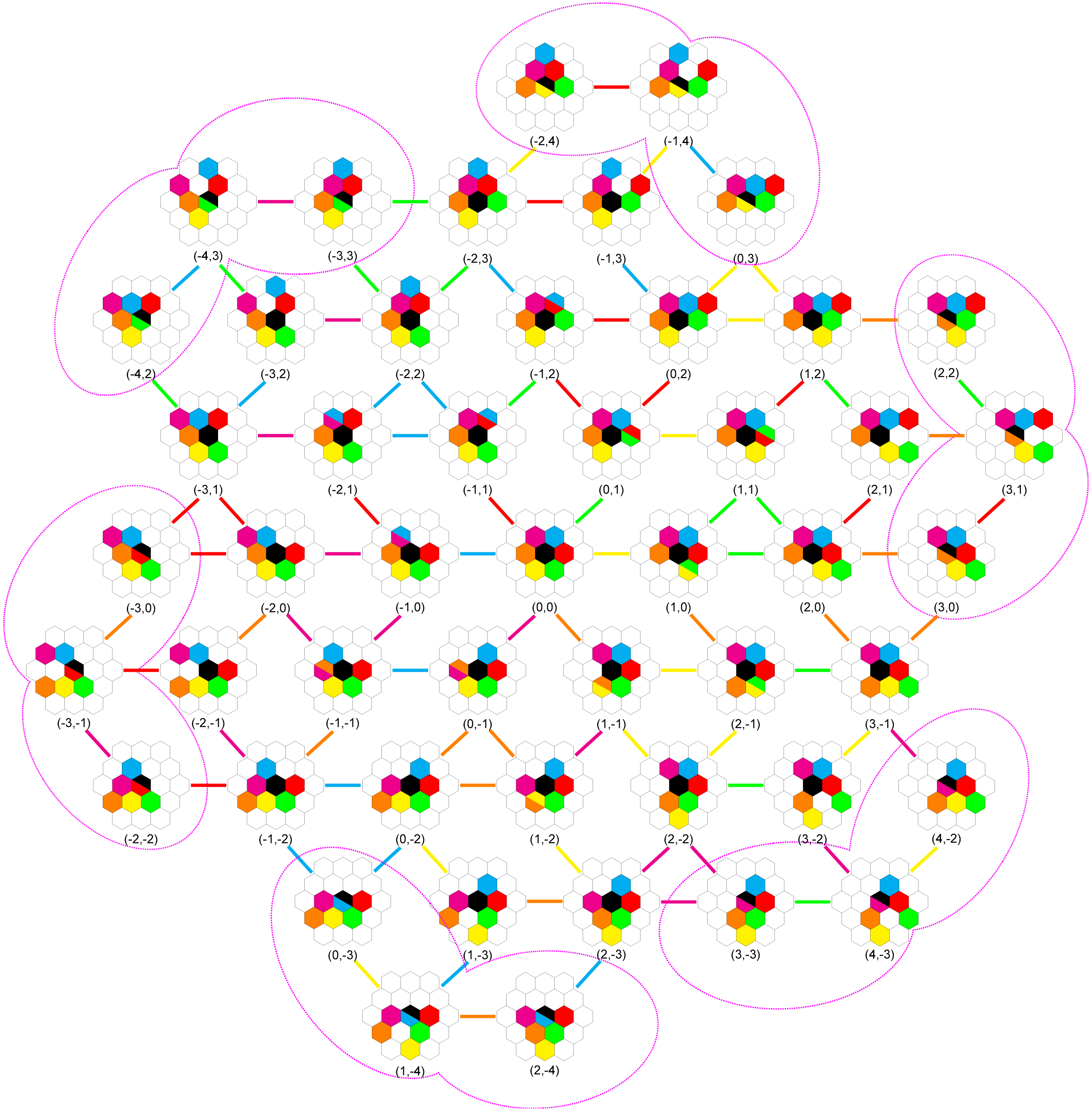
Each region bounded by critical lines is called a frame.

Proposition

For any $\kappa, \lambda \in \Lambda, \mathcal{G}_r^U(\kappa) = \mathcal{G}_r^U(\lambda)$ if and only if $\mathcal{F}(\kappa)$ and $\mathcal{F}(\lambda)$ are in the same frame.

Remainder Range Partitioning





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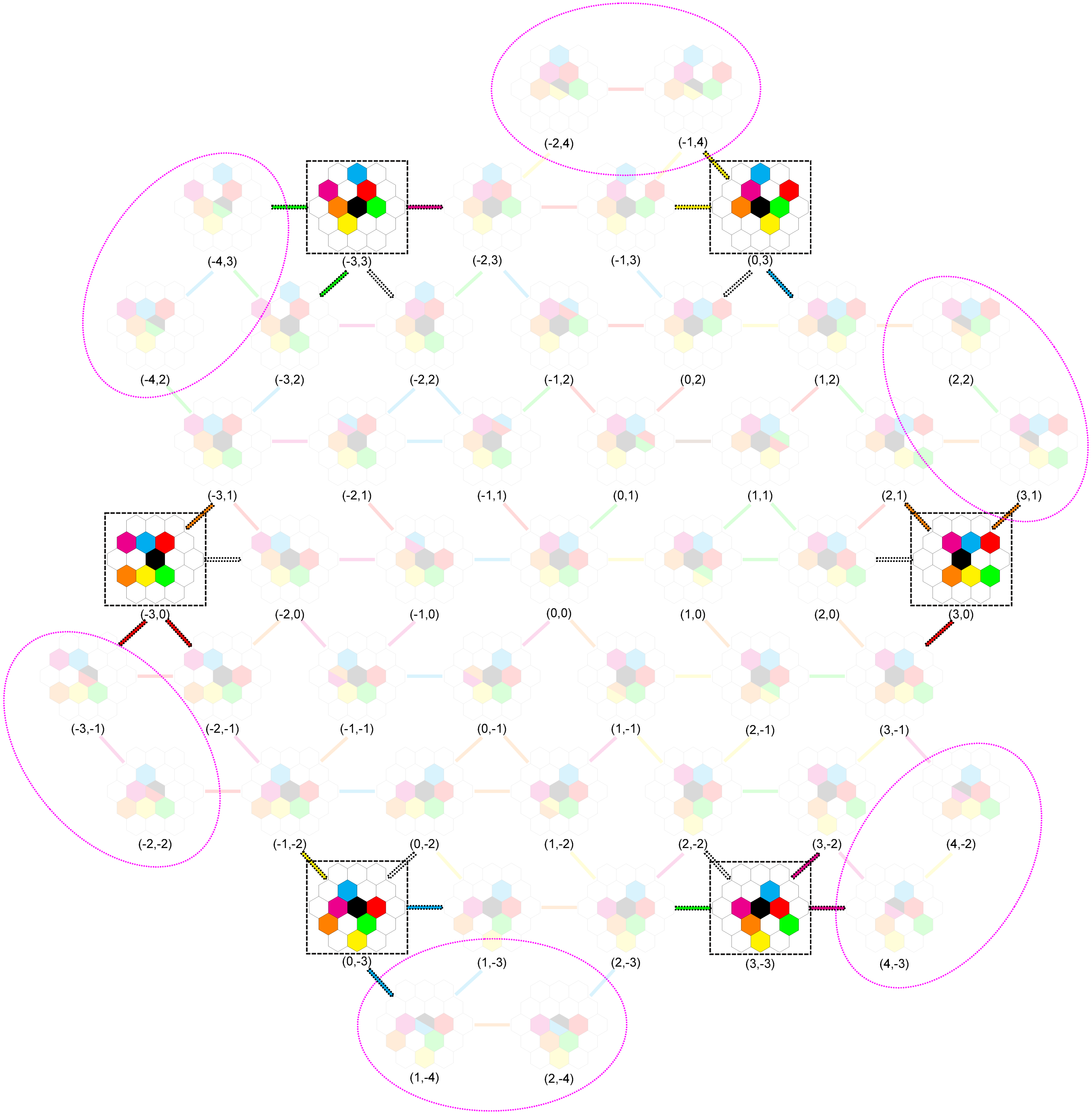
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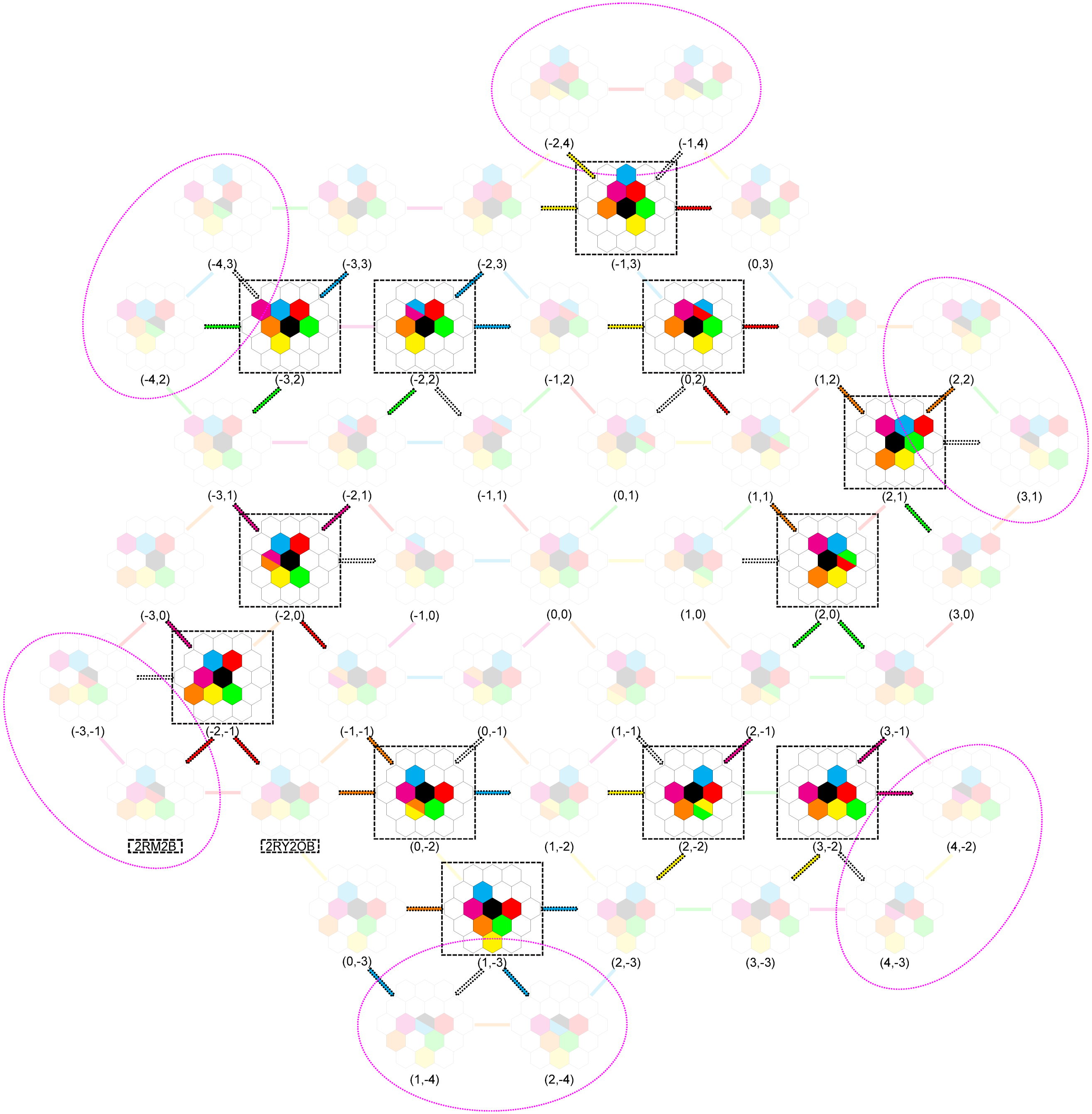

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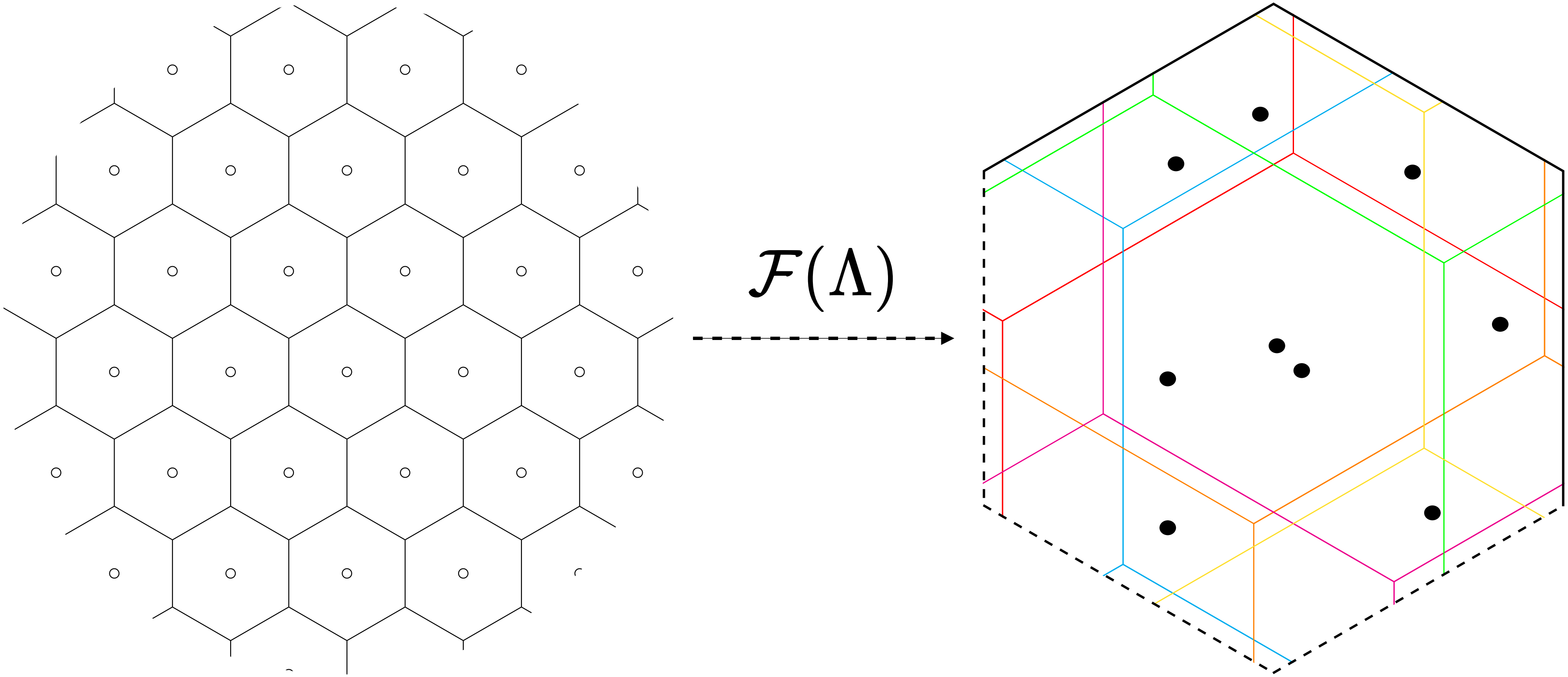
Contributions

Or extracting the pure, organic honey

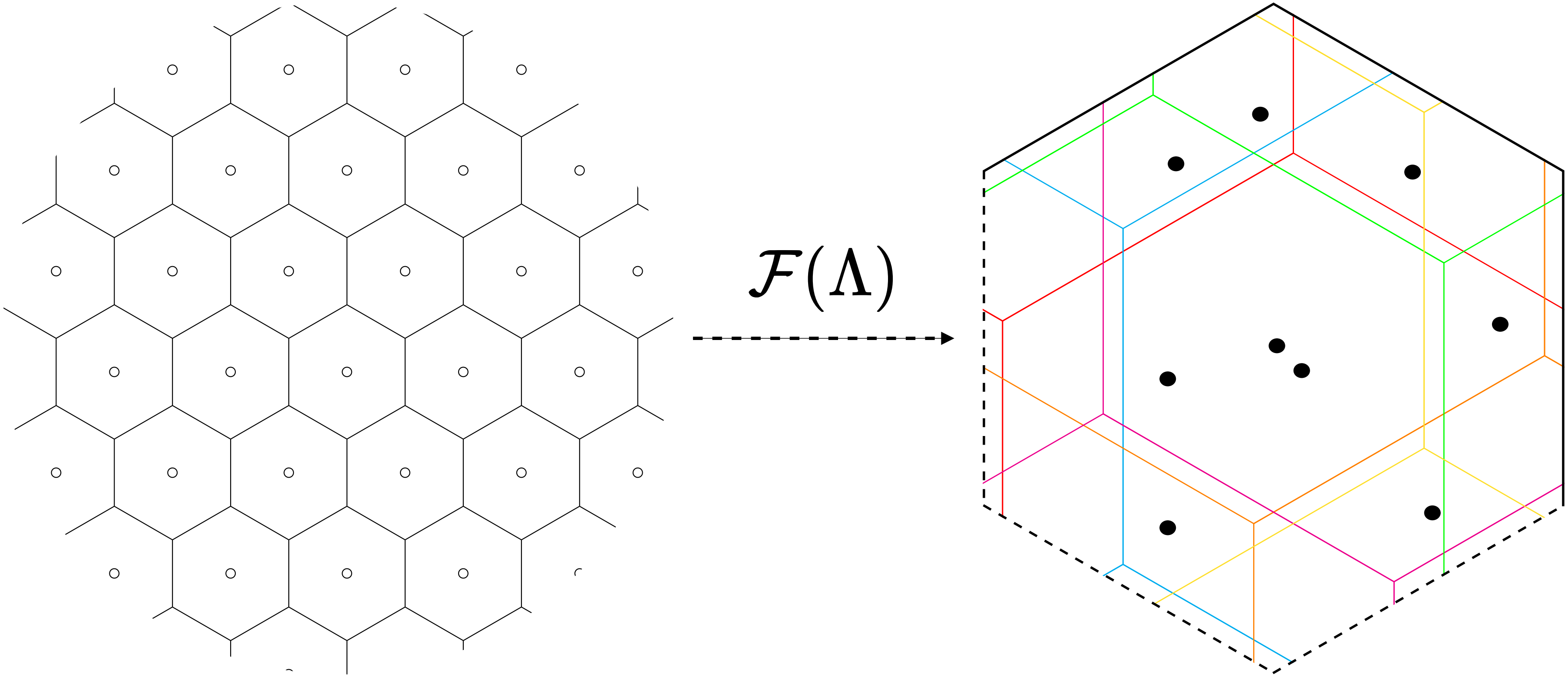
The figure comes from Wikimedia. The original comes from A practical treatise on the hive and honey-bee (public domain)

Rational Rotations



For what kind of parameters has the mapping a finite number of images?

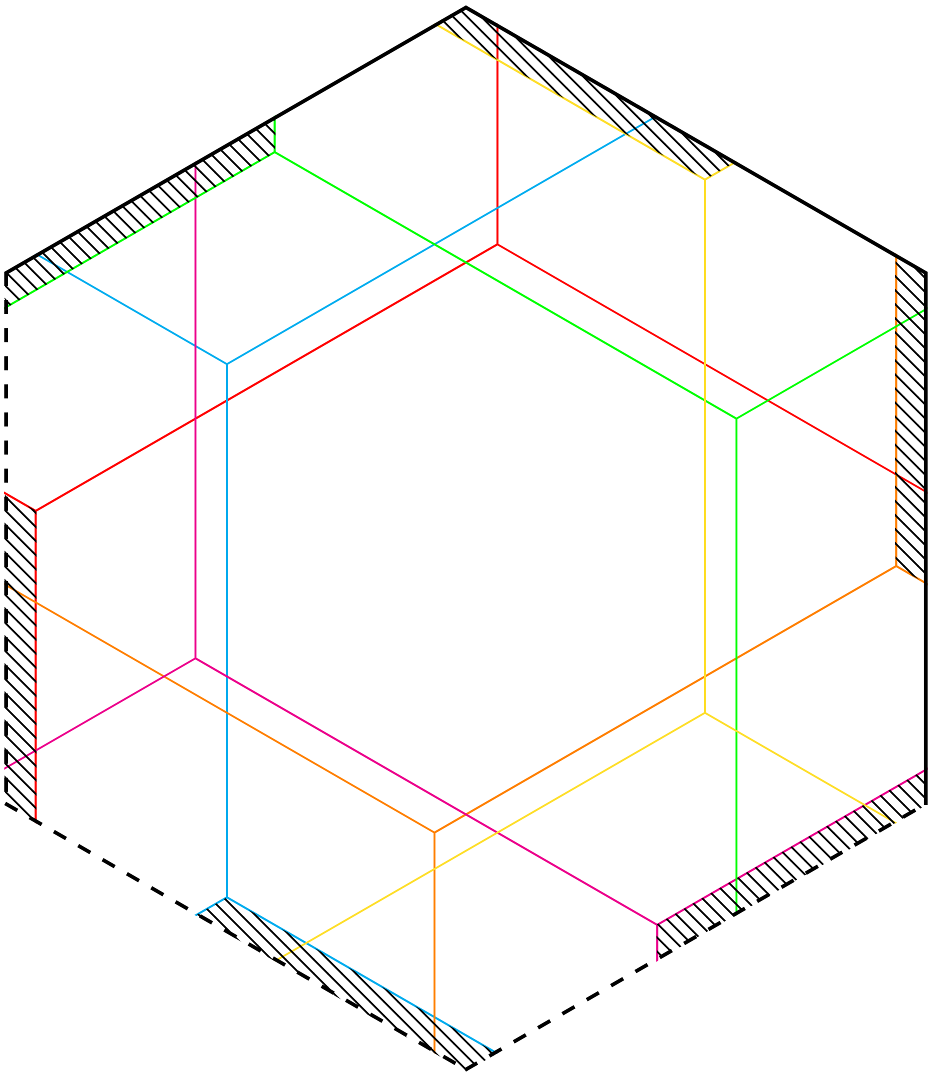
Rational Rotations



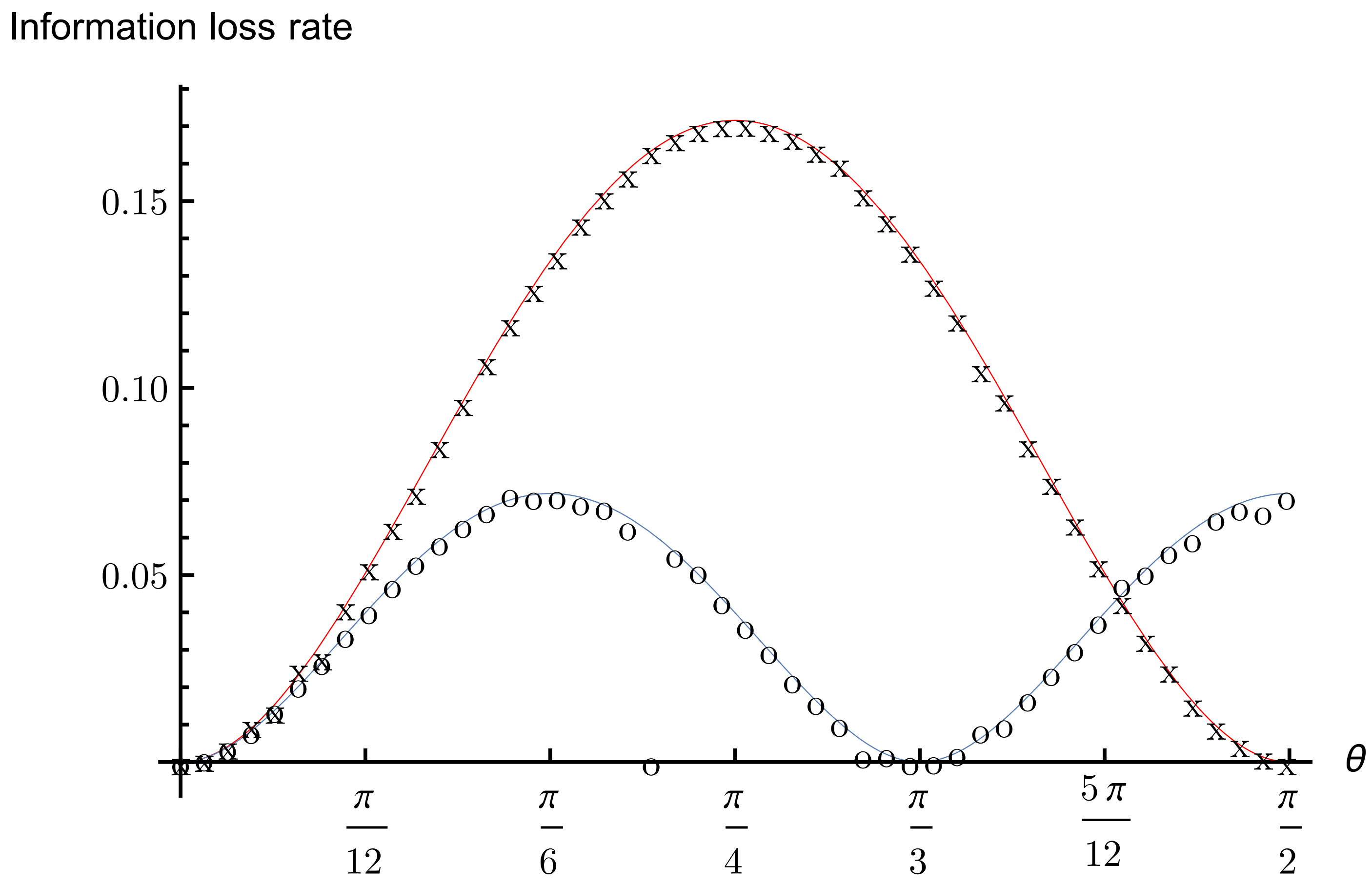
Corollary

If $\cos\theta = \frac{2a-b}{2c}$ and $\sin\theta = \frac{\sqrt{3}b}{2c}$ where $(a, b, c) \in \mathbb{Z}^3$, $\gcd(a, b, c) = 1$ and $0 < a < c < b$, then the mapping has a finite number of images.

Non-injective Digitized Rigid Motions



Loss of Information

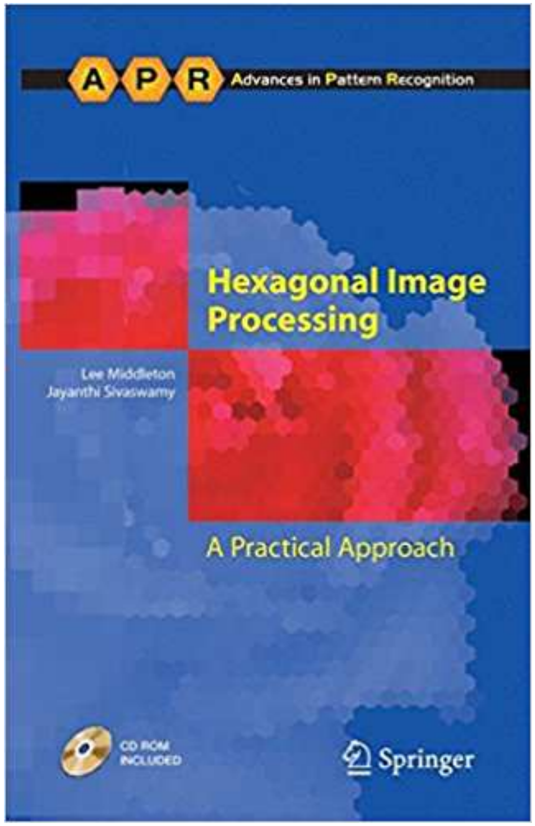


Conclusions & Perspectives

- An extension of a framework to study digitized rigid motions
- Characterization of rational rotations
- We have showed that the loss of information is relatively lower for digitized rigid motions defined on the hexagonal grid
- Our tools on BSD-3 license: https://github.com/copyme/NeighborhoodMotionMapsTools

The humble bees have been working with David Cœurjolly, Tristan Roussillon and Victor Ostromoukhov of University Lyon 1, LIRIS on some new exciting results. Stay tuned...

Homework



If you want to get into the honey business, then this book is an obligatory lecture: Middleton, Lee, and Jayanthi Sivaswamy. *Hexagonal image processing: A practical approach*. Springer Science & Business Media, 2006.