Topological alterations of 3D digital images under rigid transformations

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Agenda

Motivation

Investigation of conditions for topology preservation of 3D digital images under digitized rigid transformations.
Important in:

- Image registration
- Image classification

Related work

- Quasi-shear rotations in 2D, Andres (1996)
- Configuration induced by discrete rotations, Nouvel and Rémila (2003, 2005)
- Digital Homeomorphisms in Deformable Registration, Pierre-Louis Bazin *et al.* (2007, 2011)
- Combinatorial structure of rigid transformations, Ngo *et al.*, (2013)
- Sufficient condition for topology preservation of 2D images, Ngo *et al.*, (2014)

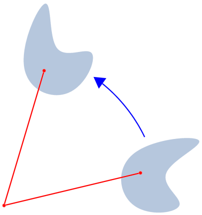
Rigid transformations in $\mathbb{R}^3$

$$\begin{array}{|lll} \mathcal{U} : \mathbb{R}^3 & \to & \mathbb{R}^3 \\ \mathbf{x} & \mapsto & \mathbf{Rx} + \mathbf{t} \end{array}$$

x – point

R – rotation matrix

t – translation vector



Properties:

- distance and angle preserving maps
- bijective
- inverse: $\mathcal{T} = \mathcal{U}^{-1}$ is also a rigid transformation

Rigid transformations in $\mathbb{Z}^3$

$$U = \mathcal{D} \circ \mathcal{U}_{|\mathbb{Z}^3}$$

$$T = \mathcal{D} \circ \mathcal{T}_{|\mathbb{Z}^3} = \mathcal{D} \circ (\mathcal{U}^{-1})_{|\mathbb{Z}^3}$$

where $\mathcal{D}$ is a digitization e.g. rounding function

Properties:

- in general, does not preserve distances and angles
- not injective
- not surjective $U(\mathbb{Z}^3) \subsetneq \mathbb{Z}^3$

Voxel statuses during rigid transformations

Definition

For a given digitized rigid transformation T, the set of transformed points in a voxel $V(\mathbf{x})$ is defined by $M(\mathbf{x}) = \{\mathbf{y} \in \mathbb{Z}^3 \mid T(\mathbf{y}) = \mathbf{x}\}$. The status of $V(\mathbf{x})$ is called *k-voxel* where, k stands for $M(\mathbf{x}$.

Property

Under rigid transformations, $k \in \{0, 1, 2, 3, 4\}$.

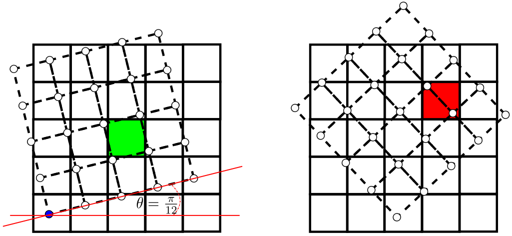
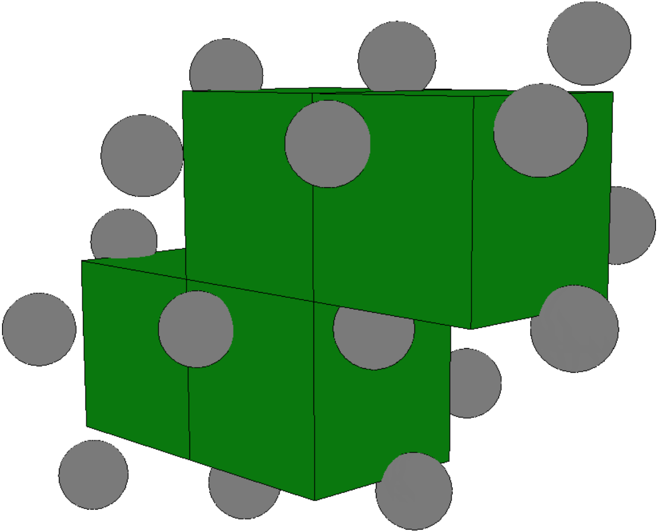
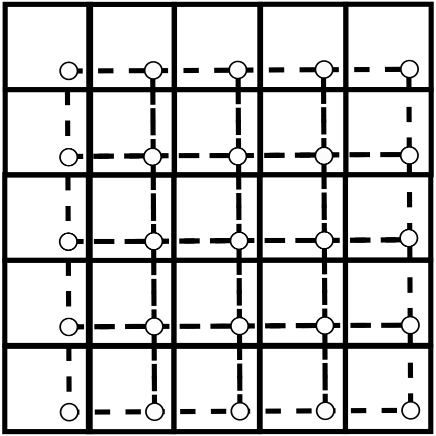


Figure : Obtained by application of $\mathcal{U}$.

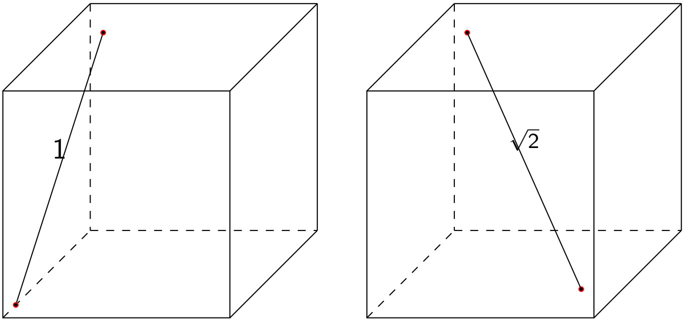
0-voxels



1-voxels



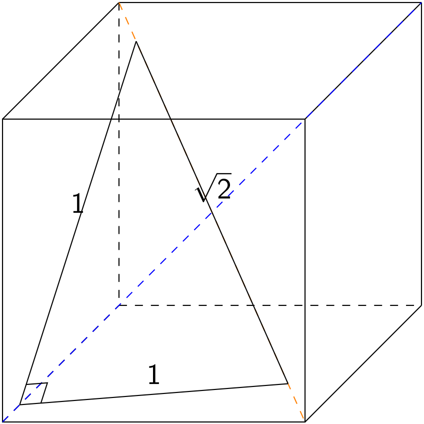
2-voxels



Property

Any couple of points $\{\mathbf{x}_1, \mathbf{x}_2\} \subseteq S$ so that $d_e(T(\mathbf{x}_2), T(\mathbf{x}_1)) \geq \sqrt{3}$ cannot fit in a voxel after any digitized rigid transformation defined on $\mathbb{Z}^3$. Here, d_e denotes Euclidean distance.

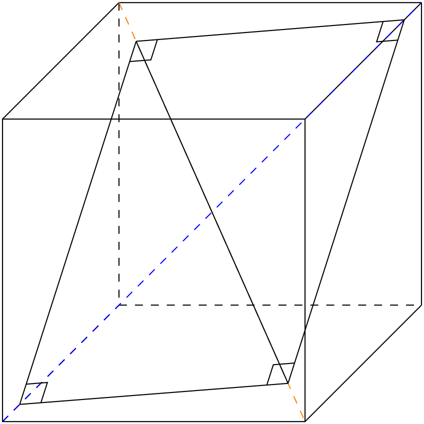
3-voxels



Property

Any triple of points $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \subseteq S$ *which form a right triangle of sides* $1, 1, \sqrt{2}$*, creates a 3-voxel under some digitized rigid transformations.*

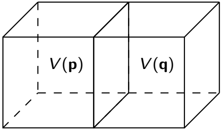
4-voxel

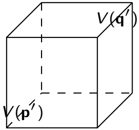


Property

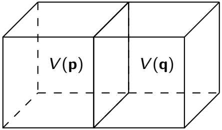
Points of 4-voxel configuration always form a unit square.

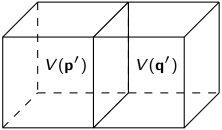
Distances induced for *6-adjacent* voxels



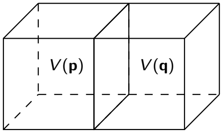


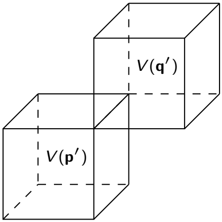
Distances induced for *6-adjacent* voxels



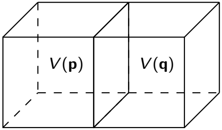


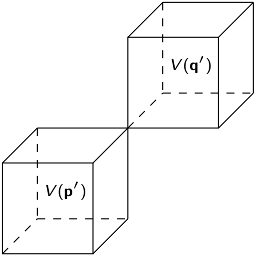
Distances induced for *6-adjacent* voxels



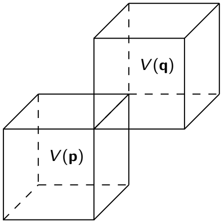


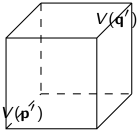
Distances induced for *6-adjacent* voxels



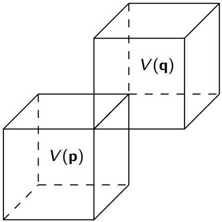


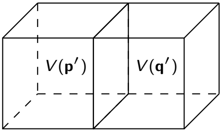
Distances induced for *18-adjacent* voxels





Distances induced for *18-adjacent* voxels





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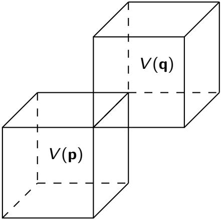

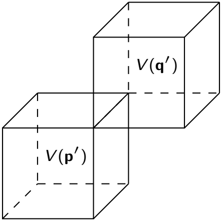

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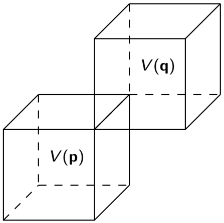

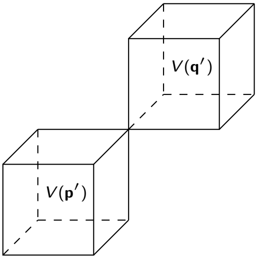

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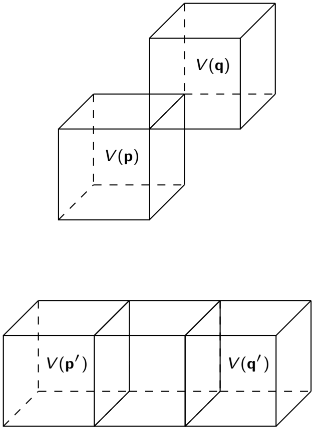

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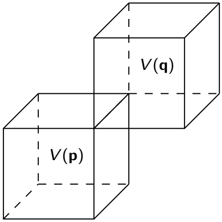
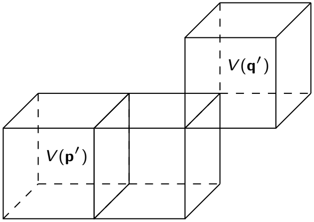
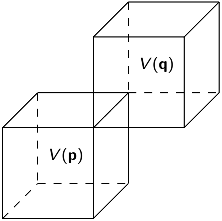
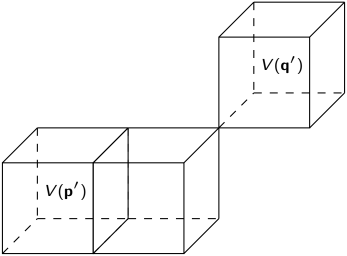


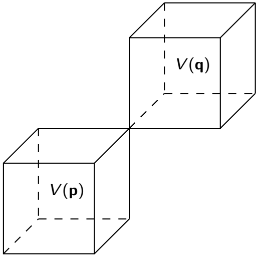


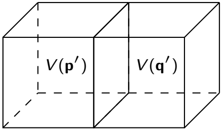
Distances induced for *18-adjacent* voxels



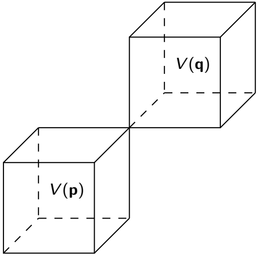


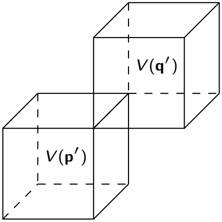
Distances induced for *26-adjacent* voxels



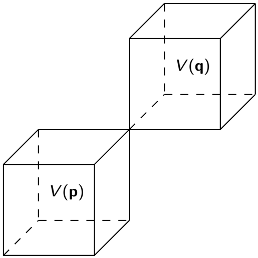


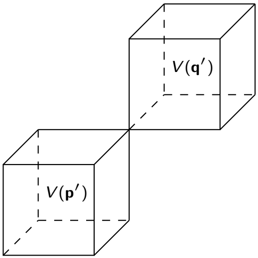
Distances induced for *26-adjacent* voxels





Distances induced for *26-adjacent* voxels





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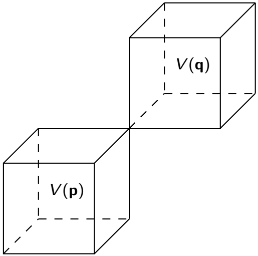
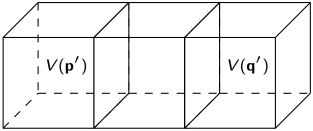
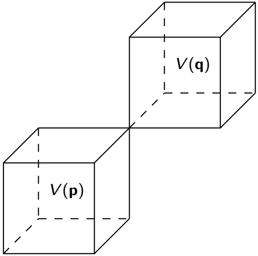
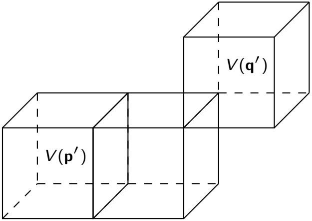




Distances induced for *26-adjacent* voxels





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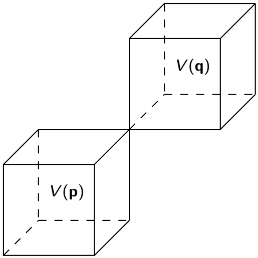

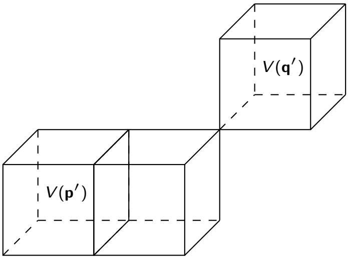

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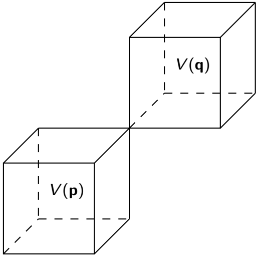
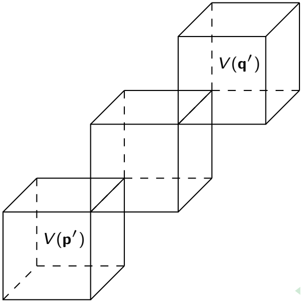
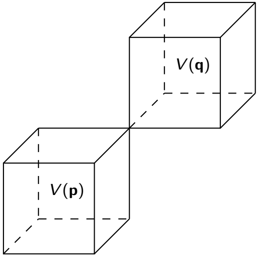
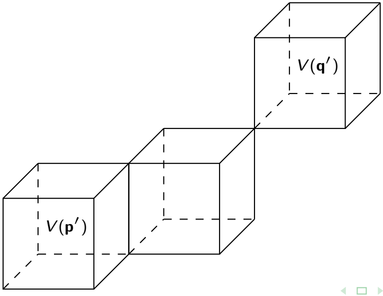




Distances induced for *26-adjacent* voxels





Formalization of distance alterations

Proposition

Let $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^3$ *be two arbitrary points with given adjacency relation* $\mathbf{p} \frown_k \mathbf{q}$ *where* $k \in \{6, 18, 26\}$*, if:*

$$d_e(\mathbf{p}, \mathbf{q}) = 1 \text{ then } d_e(T(\mathbf{p}), T(\mathbf{q})) \in \{0, 1, \sqrt{2}, \sqrt{3}\}$$

$$d_e(\mathbf{p}, \mathbf{q}) = \sqrt{2} \text{ then } d_e(T(\mathbf{p}), T(\mathbf{q})) \in \{0, 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}\}$$

$$d_e(\mathbf{p}, \mathbf{q}) = \sqrt{3} \text{ then } d_e(T(\mathbf{p}), T(\mathbf{q})) \in \{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, \sqrt{8}, 3\}$$

Condition for topology preservation

- Pierre-Louis Bazin, Lotta Maria Ellingsen, and Dzung L. Pham.

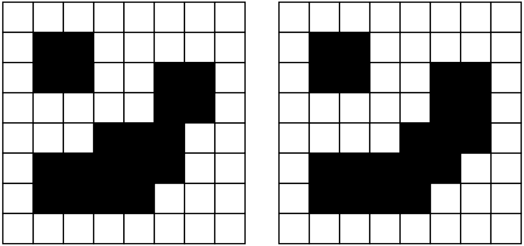
  Digital homeomorphisms in deformable registration.
  In *IPMI*, volume 4584 of *Lecture Notes in Computer Science*, pages 211–222. Springer, 2007
- Pierre-Louis Bazin, Navid Shiee, Lotta M. Ellingsen, Jerry L. Prince,

  and Dzung L. Pham. *Digital topology in brain image segmentation and registration*, volume 1, chapter 12, pages 339–375.
  Springer, 2011
- P. Ngo, N. Passat, Y. Kenmochi, and H. Talbot. Topology-preserving rigid transformation of 2D digital images.
  IEEE Transactions on Image Processing, 23(2):885–897, 2014

Topology preservation, Bazin *et al.*, 2007, 2011

Theorem

Rigid transformation is guaranteed to preserve the topology of connected component C if and only if the distance between any two points outside C such that the line between them intersects C is strictly higher than $\sqrt{3}$ for 3D or $\sqrt{2}$ for 2D.

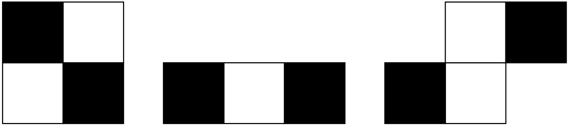


Definition

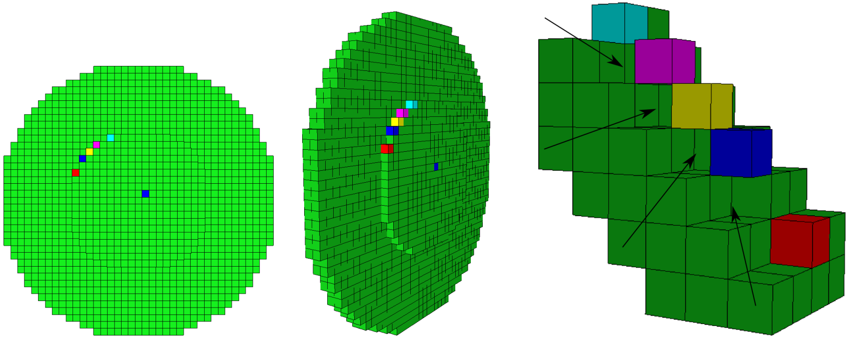
Let I be *non-singular*, *well-composed*, *binary* image. Let $v \in \{0, 1\}$. We say that I is *v-regular* if for any $\mathbf{p}, \mathbf{q} \in I^{-1}(\{v\})$, we have

$$(\mathbf{p} \frown_4 \mathbf{q}) \Rightarrow (\exists \boxplus \subseteq I^{-1}(\{v\}), \mathbf{p}, \mathbf{q} \in \boxplus)$$

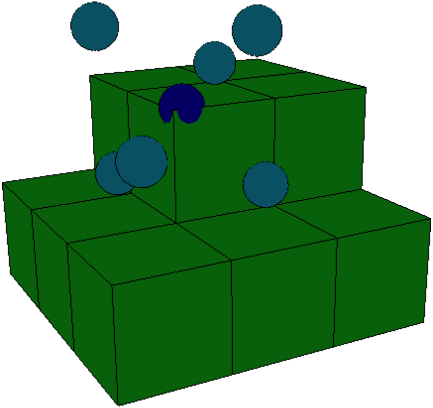
where $\boxplus = \{x, x+1\} \times \{y, y+1\}$, for $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^2$. We say that I is *regular* if it is both *0-* and *1-regular*.



Difficulties with topology of 3D images



Difficulties with topology of 3D images



Conclusion and perspectives

- Potential loss of information induced by voxels of high statuses.
- Possible alteration of connected components, such as merging or splitting.
- Notations of regular images cannot be extended to 3D.

Better understanding of adjacency structure of image under digitized rigid transformations should lead to efficient techniques of topology reparation.

Thank you for your attention!