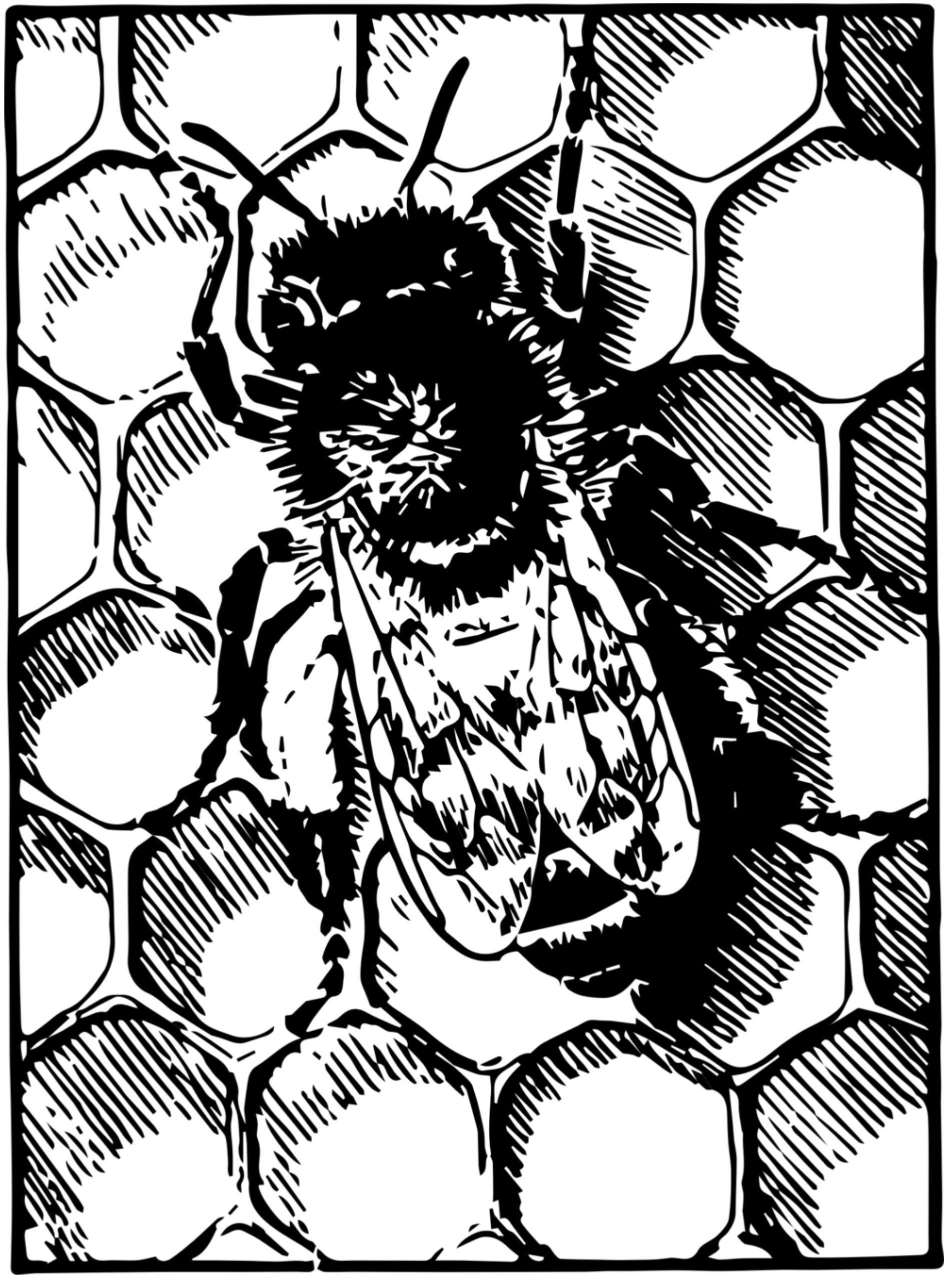
Honeycomb Geometry

Rigid Motions on the Hexagonal Grid

by Kacper Pluta, Pascal Romon,

Yukiko Kenmochi and Nicolas Passat



The figure comes from "Insects The Yearbook of Agriculture 1952" United States Dept. of Agriculture." Published by the US Government Printing Office. Deemed to be in the Public Domain under US Law.

Motivations

Digitized rigid motions defined on the square grid are burdened with an incompatibility between rotations and the geometry of the grid.

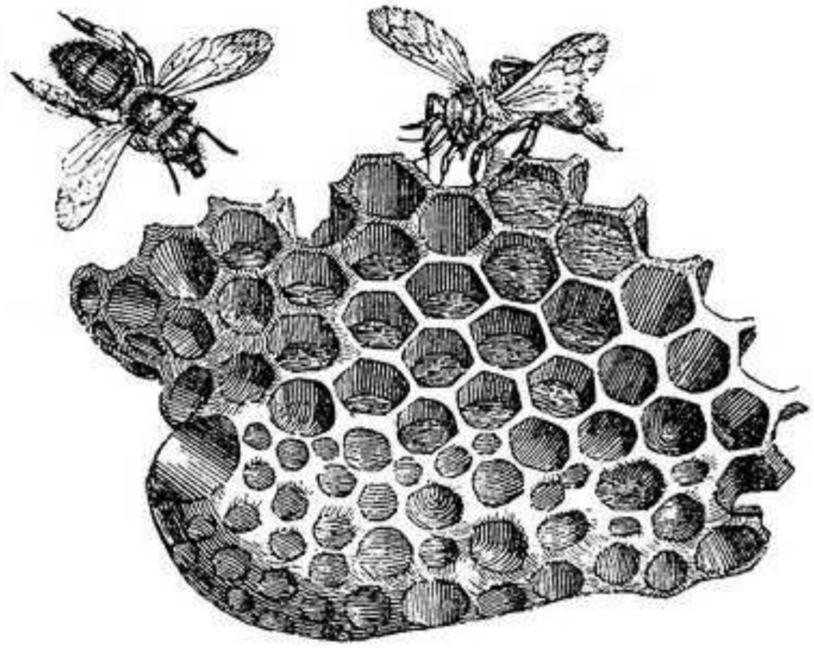
Agenda

- Introduction to the Bees' Point of View
- Quick Introduction to Rigid Motions
- Neighborhood Motion Maps
- Contributions
- Conclusions & Perspectives

The beehive figure's source and author unknown (if you recognize it, please let me know). The image of the bee comes from http://karenswhimsy.com/public-domain-images (public domain)

Introduction to the Bees' Point of View

Or why bees are right



The figure comes from http://thegraphicsfairy.com/vintage-clip-art-bees-with-honeycomb

Pros and Cons

Square grid

- \+ Memory addressing
- \+ Sampling is easy to define
- \- Sampling is not optimal (ask bees)
- \- Neighbors are not equidistant
- \- Connectivity paradox

Hexagonal grid

- \+ Uniform connectivity
- \+ Equidistant neighbors
- \+ Sampling is optimal
- \- Memory addressing is not trivial
- \- Sampling is difficult to define

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Hexagonal grid

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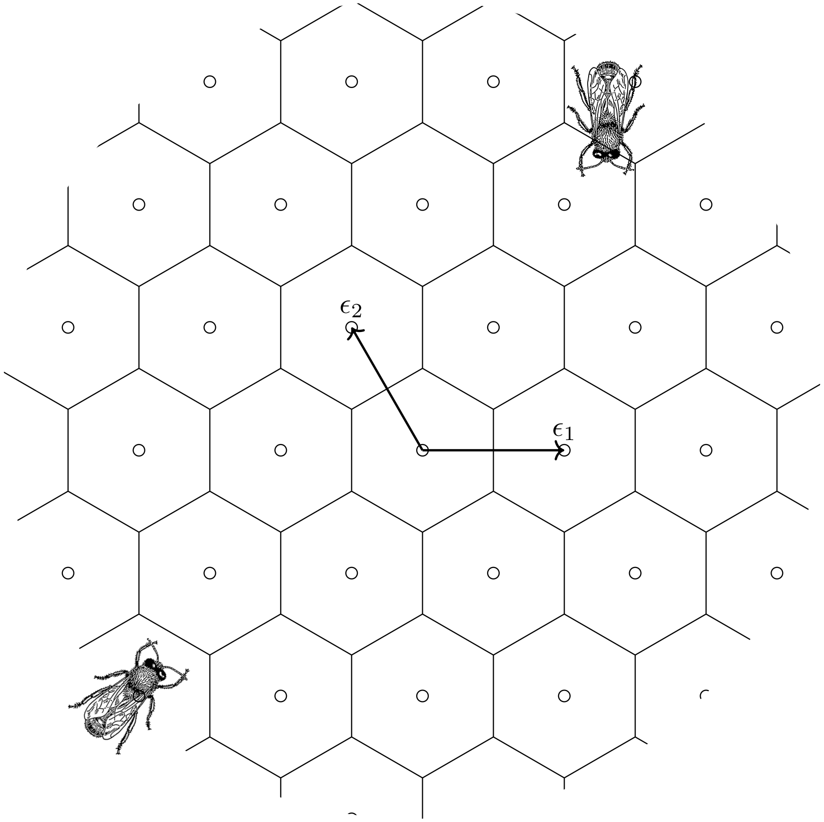
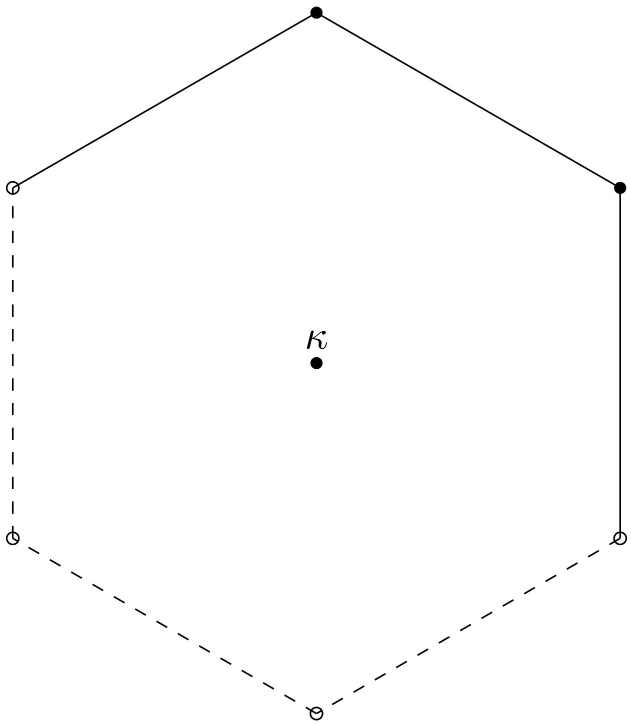
Hexagonal Grid



The hexagonal lattice: $\Lambda = \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2$ and the hexagonal grid $\mathcal{H}$

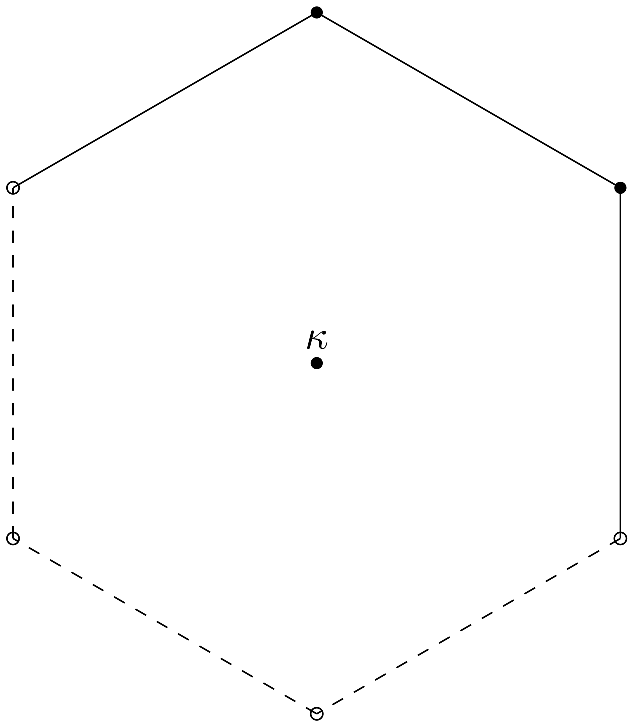
The figure by Pearson Scott Foresman, Wikimedia.

Digitization Model



The digitization cell of $\boldsymbol{\kappa}$ denoted by $\mathcal{C}(\kappa)$.

Digitization Model



The digitization operator is defined as

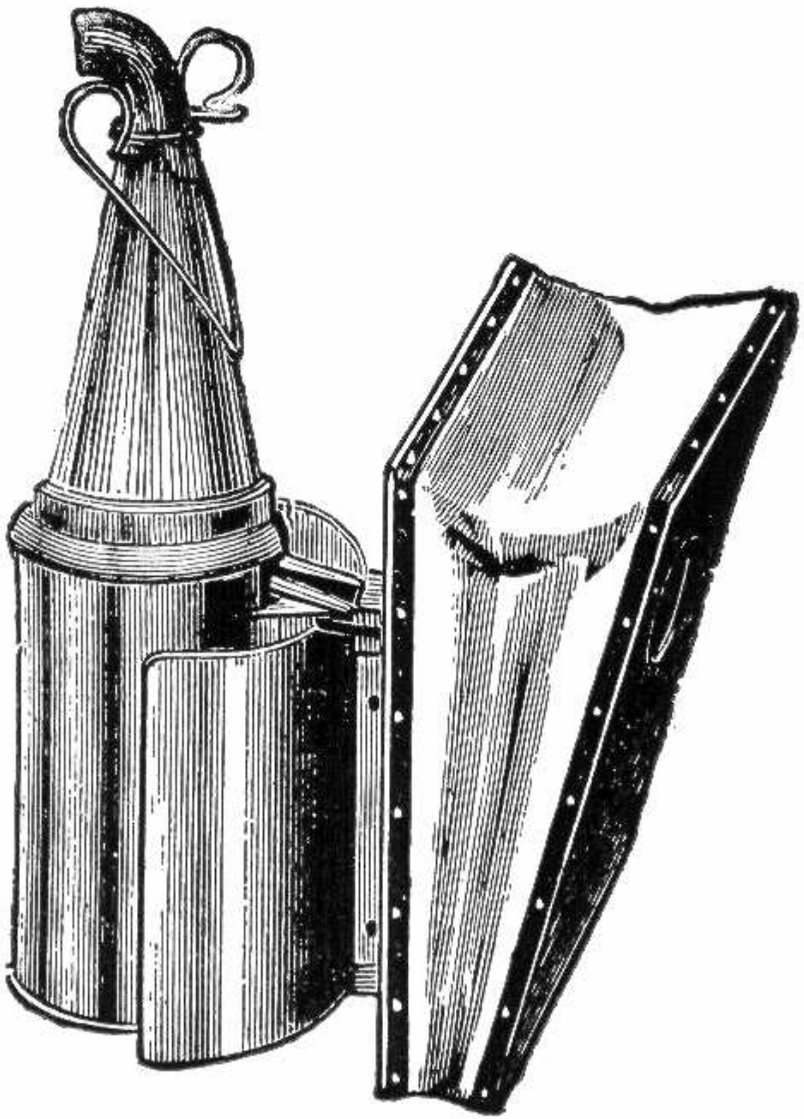
$$\mathcal{D} : \mathbb{R}^2 \to \mathbf{\Lambda}$$

such that

$$\forall \mathbf{x} \in \mathbb{R}^2, \exists! \mathcal{D}(\mathbf{x}) \in \mathbf{\Lambda} \text{ and } \mathbf{x} \in \mathcal{C}(\mathcal{D}(\mathbf{x})).$$

Quick Lesson on Rigid Motions

Or how to become a beekeeper. Part I - Equipment



The figure comes from Wikimedia. Original source *The New Student's Reference Work* *(public domain)*

Rigid Motions on $\mathbb{R}^2$

$$\mathcal{U} : \begin{array}{ccc} \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\ \mathbf{x} & \mapsto & \mathbf{R}\mathbf{x} + \mathbf{t} \end{array}$$

$\mathbf{R}$ - rotation matrix

$\mathbf{t}$ - translation vector

Properties

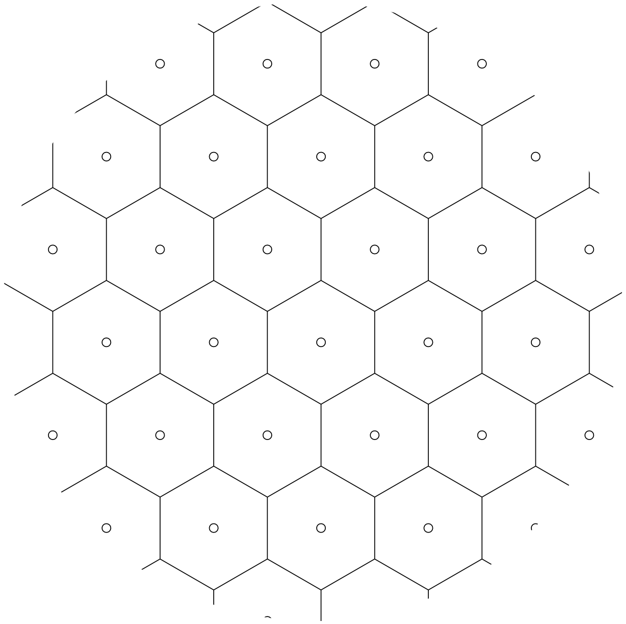
- Isometry
- Bijective

Rigid Motions on Λ

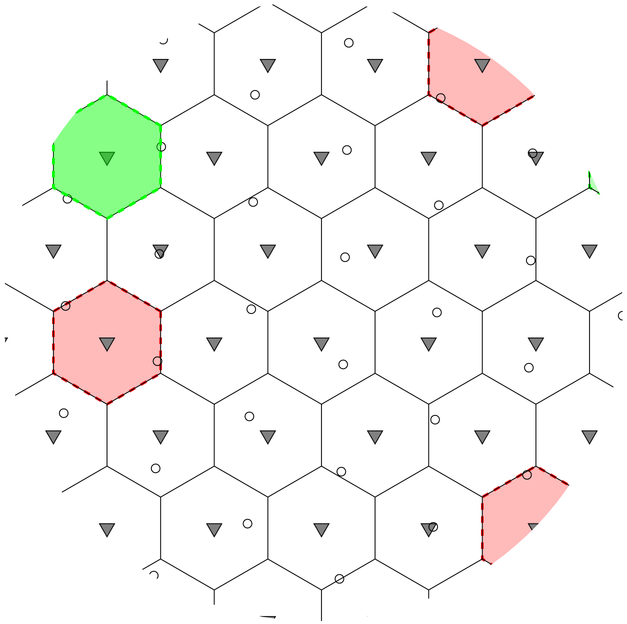
$$U = \mathcal{D} \circ \mathcal{U}_{|\Lambda}$$

Properties

- Do not preserve distances
- Non-injective
- Non-surjective



$\mathcal{U}(\Lambda)$



Related Studies

- Nouvel, B., Rémila, E.: On colorations induced by discrete rotations. In: DGCI, Proceedings. *Volume 2886 of Lecture Notes in Computer Science*., Springer (2003) 174–183
- Pluta, K., Romon, P., Kenmochi, Y., Passat, N.: Bijective digitized rigid motions on subsets of the plane. *Journal of Mathematical Imaging and Vision* (2017)

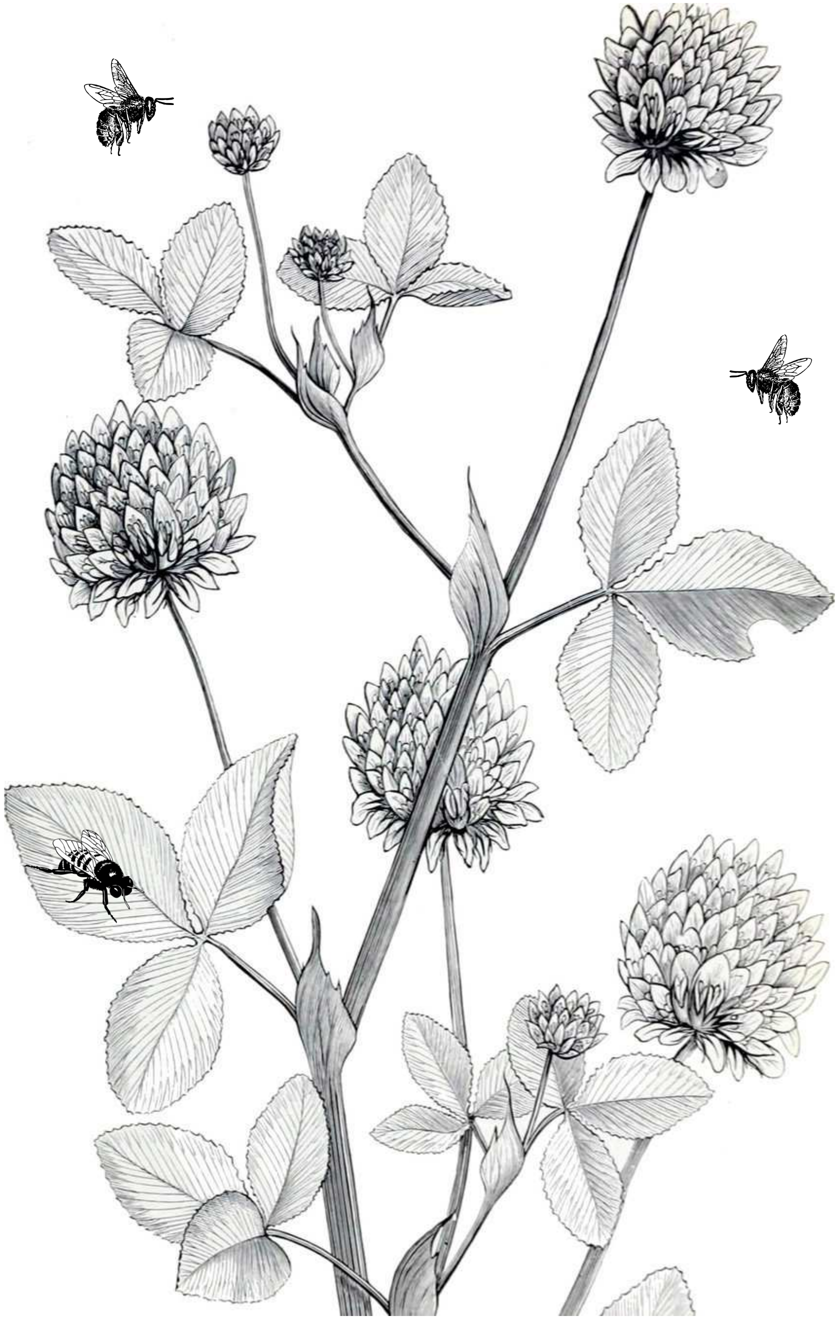
Contributions in Short

Pure extracted honey

- Extension of the former framework to the hexagonal grid
- Comparison of the loss of information between the hexagonal and square grids
- Complete set of neighborhood motion maps
- Source code of a tool to study digitized rigid motions on the hexagonal grid

Neighborhood Motion Maps

Or a manual of instructions in apiculture

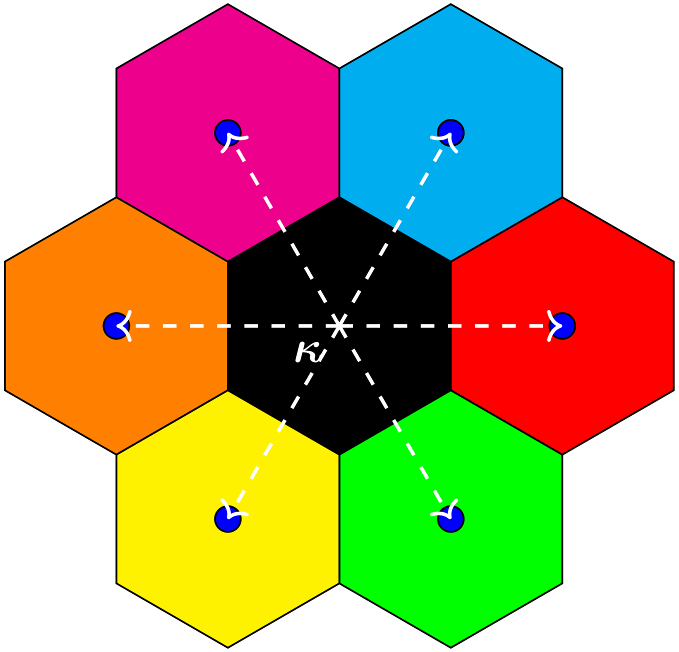


The figure comes from Wikimedia. The original source The honey bee: a manual of instruction in apiculture (public domain)

Neighborhood

The *neighborhood* of $\boldsymbol{\kappa} \in \Lambda$ (of squared radius $r \in \mathbb{R}_+$):

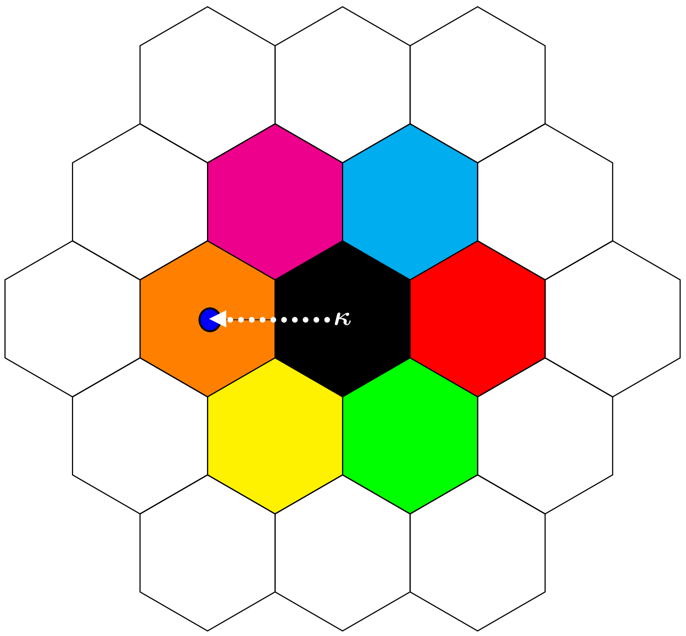
$$\mathcal{N}_r(\boldsymbol{\kappa}) = \{\boldsymbol{\kappa} + \delta \in \Lambda \mid \|\delta\|^2 \leq r\}$$

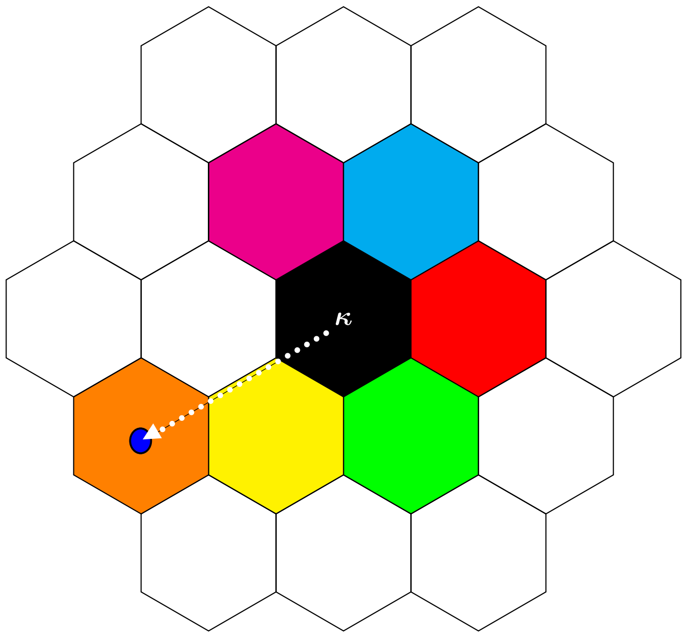


Neighborhood Motion Maps

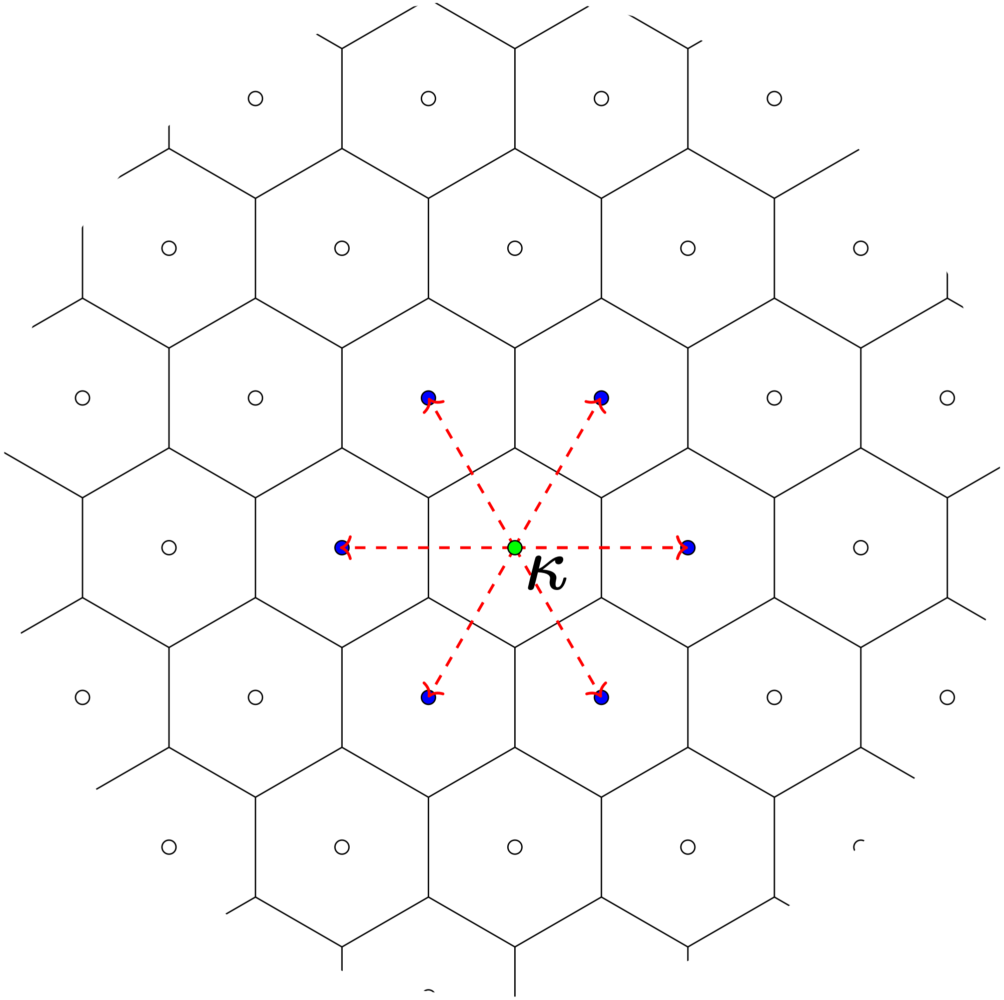
The *neighborhood motion map* of $\kappa \in \Lambda$ for given a rigid motion U and $r \in \mathbb{R}_+$

$$\left| \begin{array}{cccc} \mathcal{G}_r^U & : & \mathcal{N}_r(0) & \to & \mathcal{N}_{r'}(0) \\ & & \delta & \mapsto & U(\kappa+\delta) - U(\kappa). \end{array} \right.$$

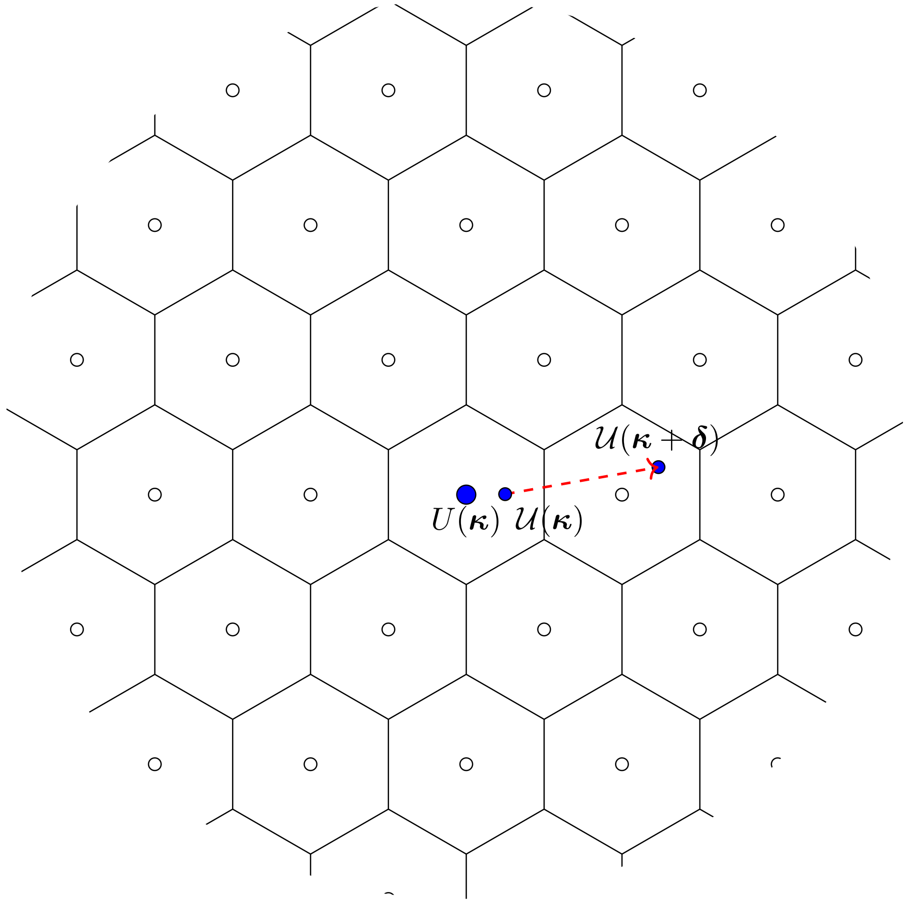




Remainder Map step-by-step

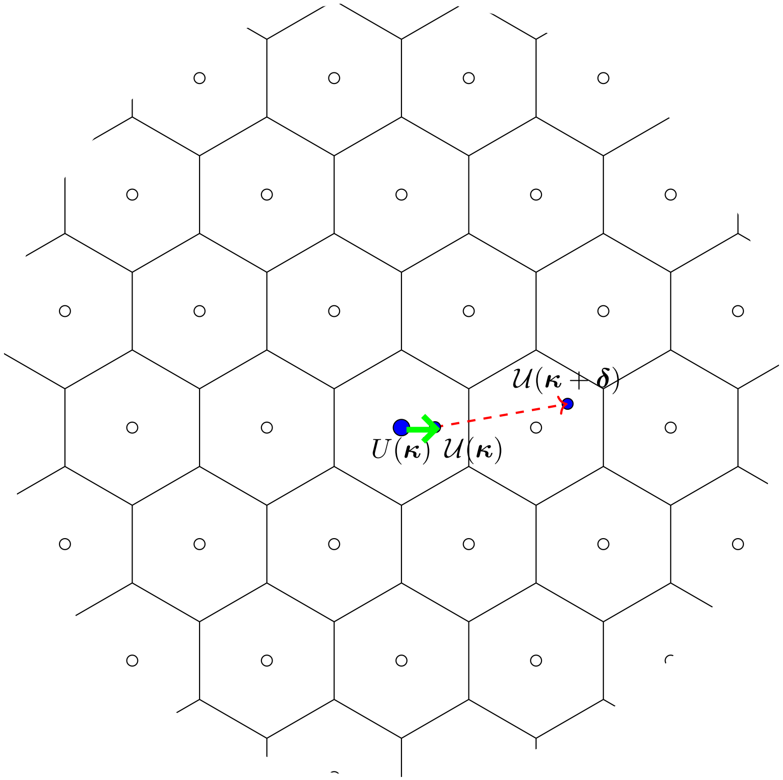


Remainder Map step-by-step



$$\mathcal{U}(\kappa+\delta) = \mathbf{R}\delta + \mathcal{U}(\kappa)$$

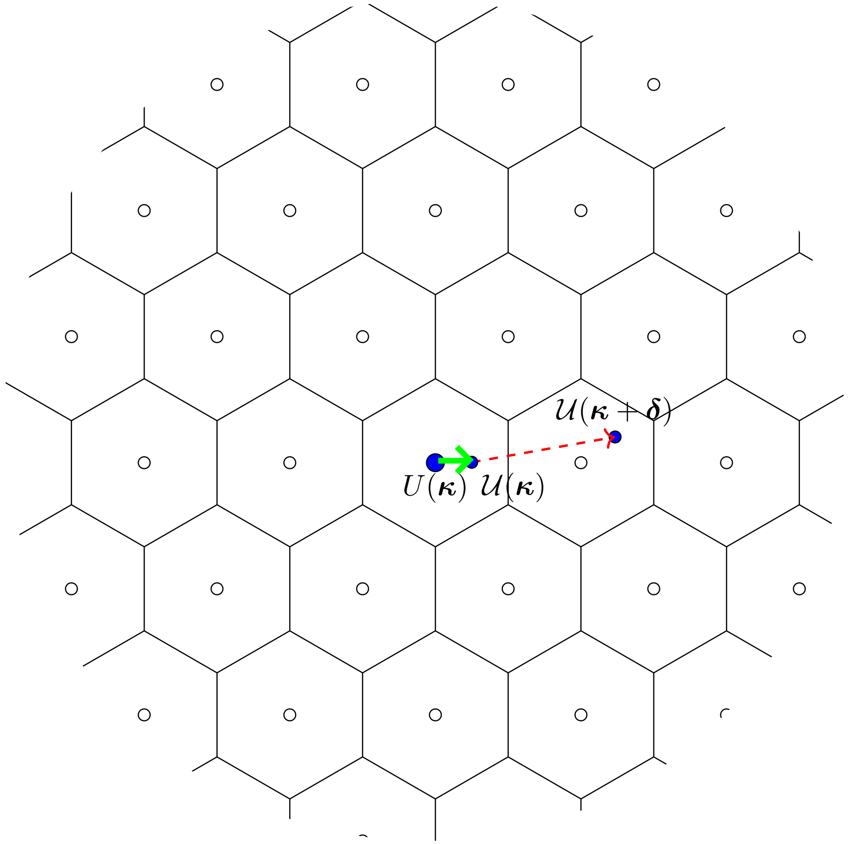
Remainder Map step-by-step



Without loss of generality, $U(\kappa)$ is the origin, and then

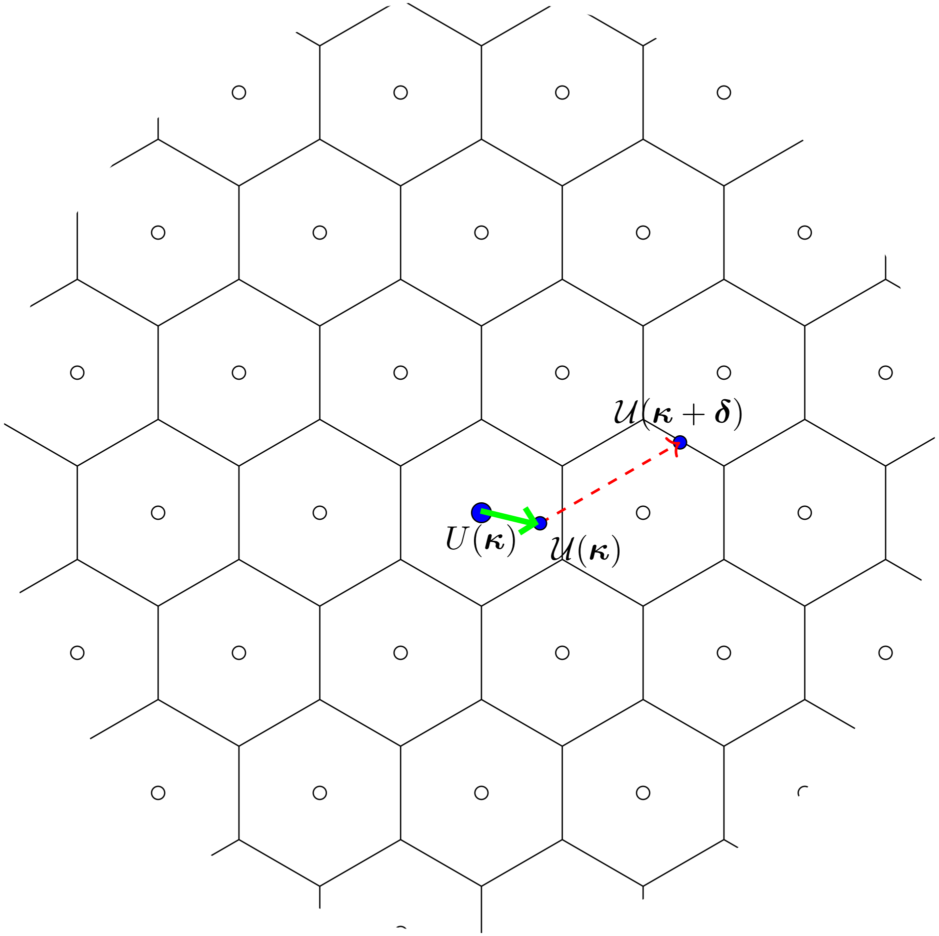
$$\mathcal{U}(\delta) = \mathbf{R}\delta + \mathcal{U}(\kappa) - U(\kappa)$$

Remainder Map step-by-step

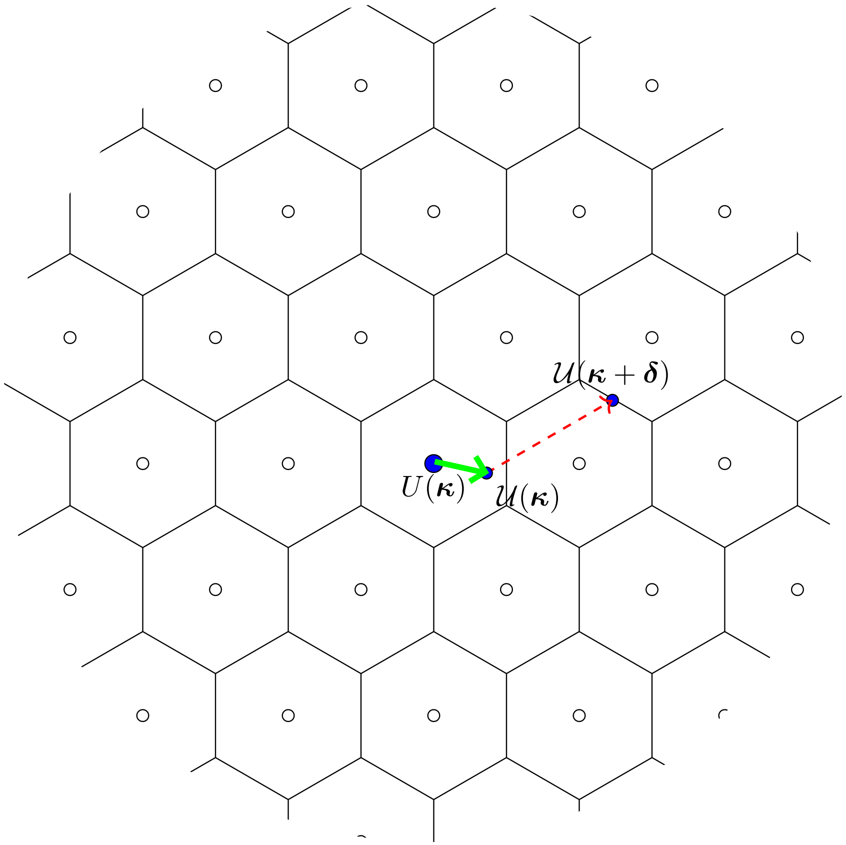


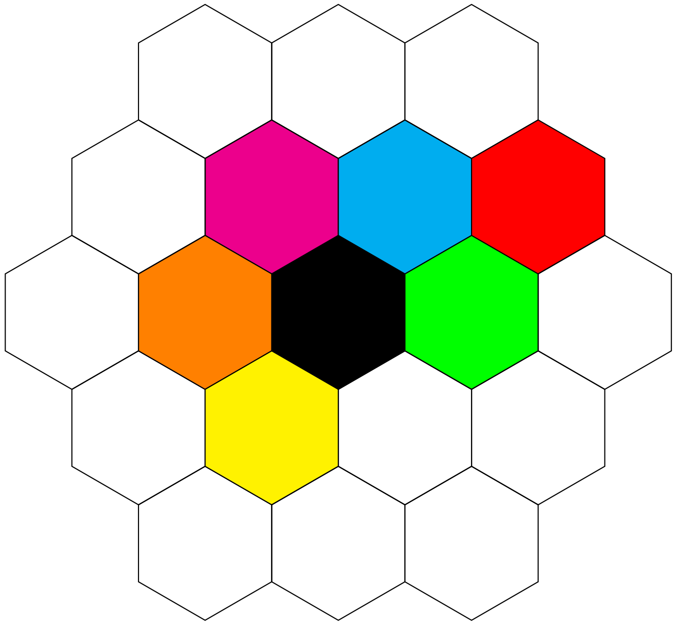
The remainder map defined as $\mathcal{F}(\boldsymbol{\kappa}) = \mathcal{U}(\boldsymbol{\kappa}) - U(\boldsymbol{\kappa}) \in \mathcal{C}(\mathbf{0})$ where the range $\mathcal{C}(\mathbf{0})$ is called the remainder range.

Remainder Map and Critical Rigid Motions

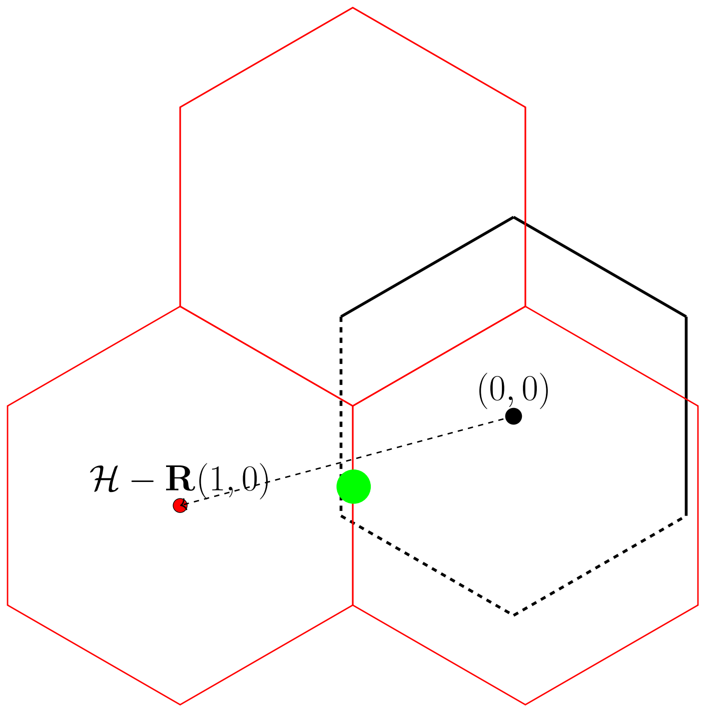


Remainder Map and Critical Rigid Motions



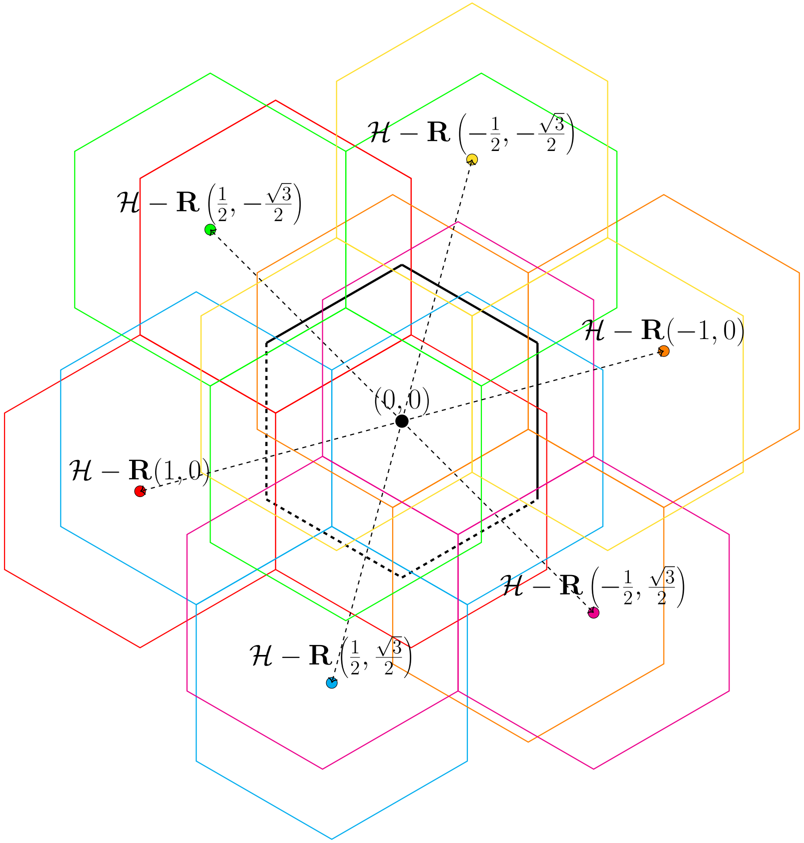


Remainder Map and Critical Rigid Motions



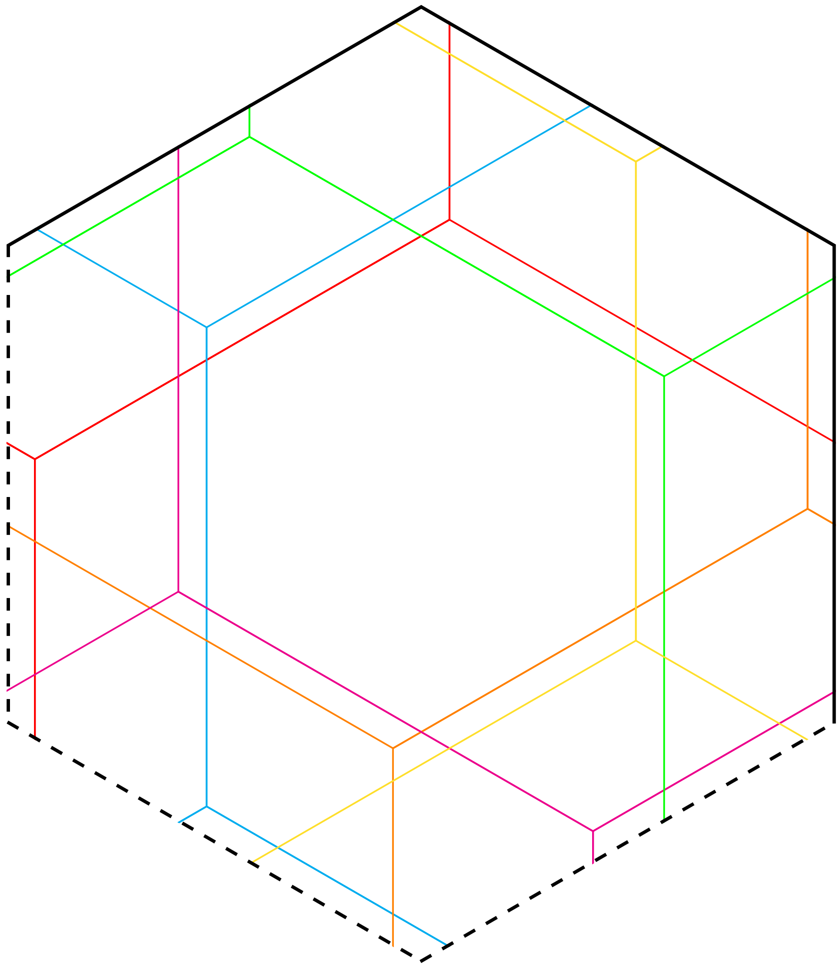
Such critical cases can be observed via the relative positions of $\mathcal{F}(\kappa)$, and are formulated as the translation $\mathcal{H} - \mathbf{R}\delta$. That is to say $\mathcal{C}(\mathbf{0}) \cap (\mathcal{H} - \mathbf{R}\delta)$.

Critical line segments



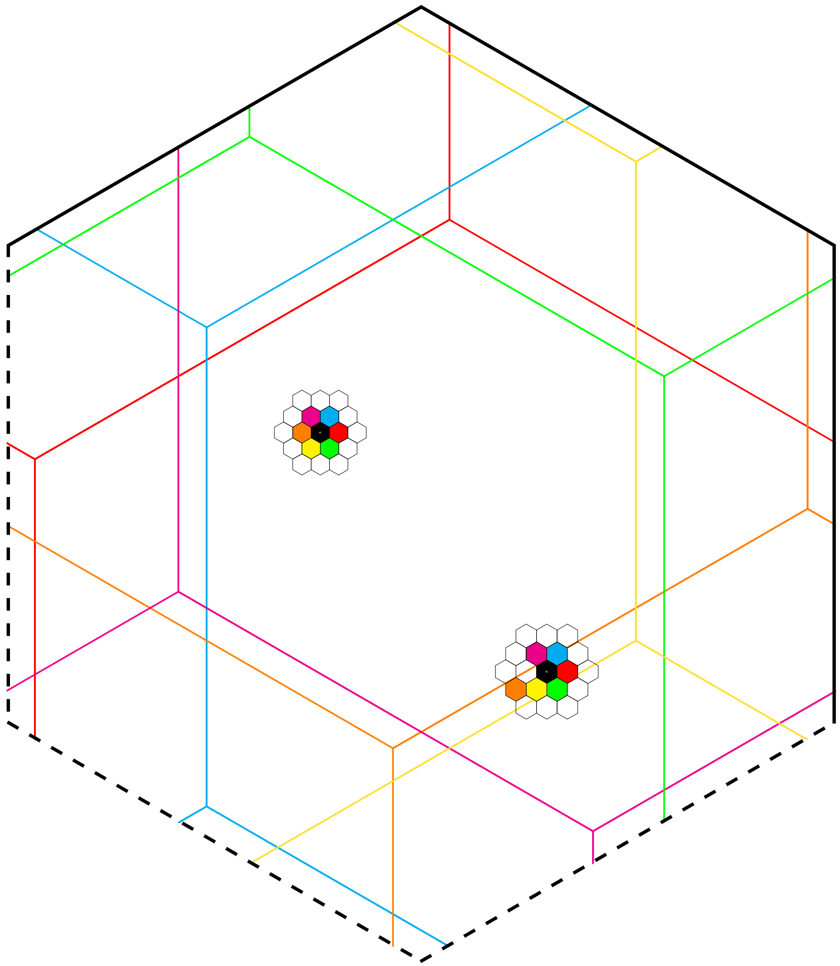
$$\mathscr{H} = \bigcup_{\delta \in \mathcal{N}_r(\mathbf{0})} (\mathcal{H} - \mathbf{R}\delta) \cap \mathcal{C}(\mathbf{0})$$

Frames



Each region bounded by the critical line segments is called a frame.

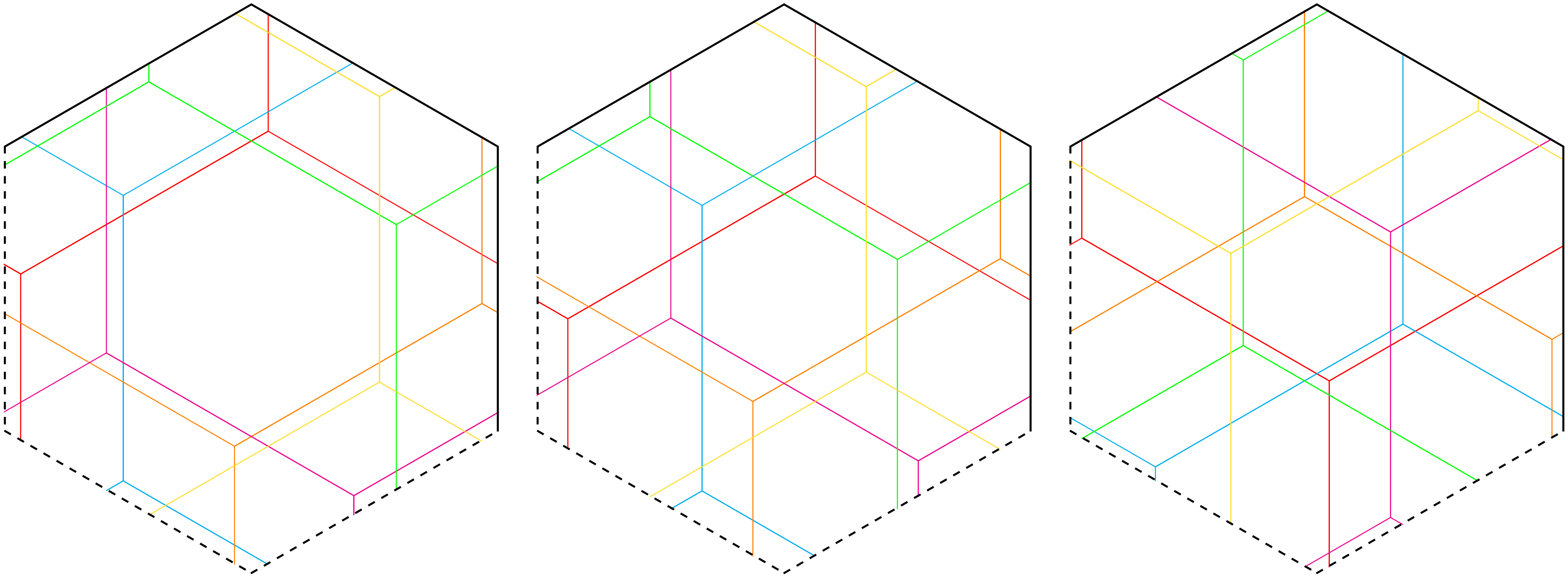
Frames



Proposition

For any $\kappa, \lambda \in \Lambda, \mathcal{G}_r^U(\kappa) = \mathcal{G}_r^U(\lambda)$ if and only if $\mathcal{F}(\kappa)$ and $\mathcal{F}(\lambda)$ are in the same frame.

Remainder Range Partitioning

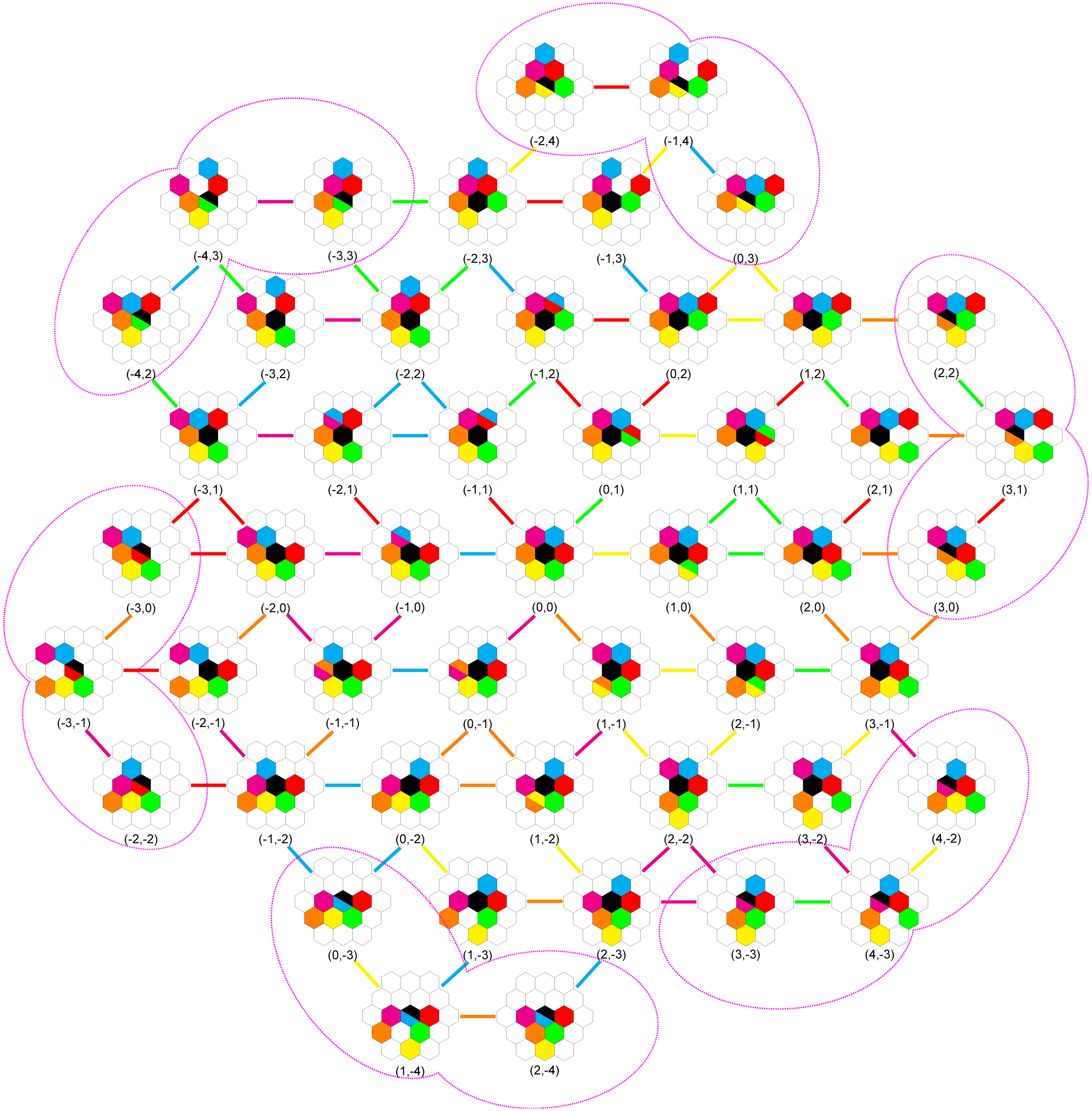


At most 49 frames per partitioning.

Contributions

Or extracting the pure, organic honey

The figure comes from Wikimedia. The original comes from A practical treatise on the hive and honey-bee (public domain)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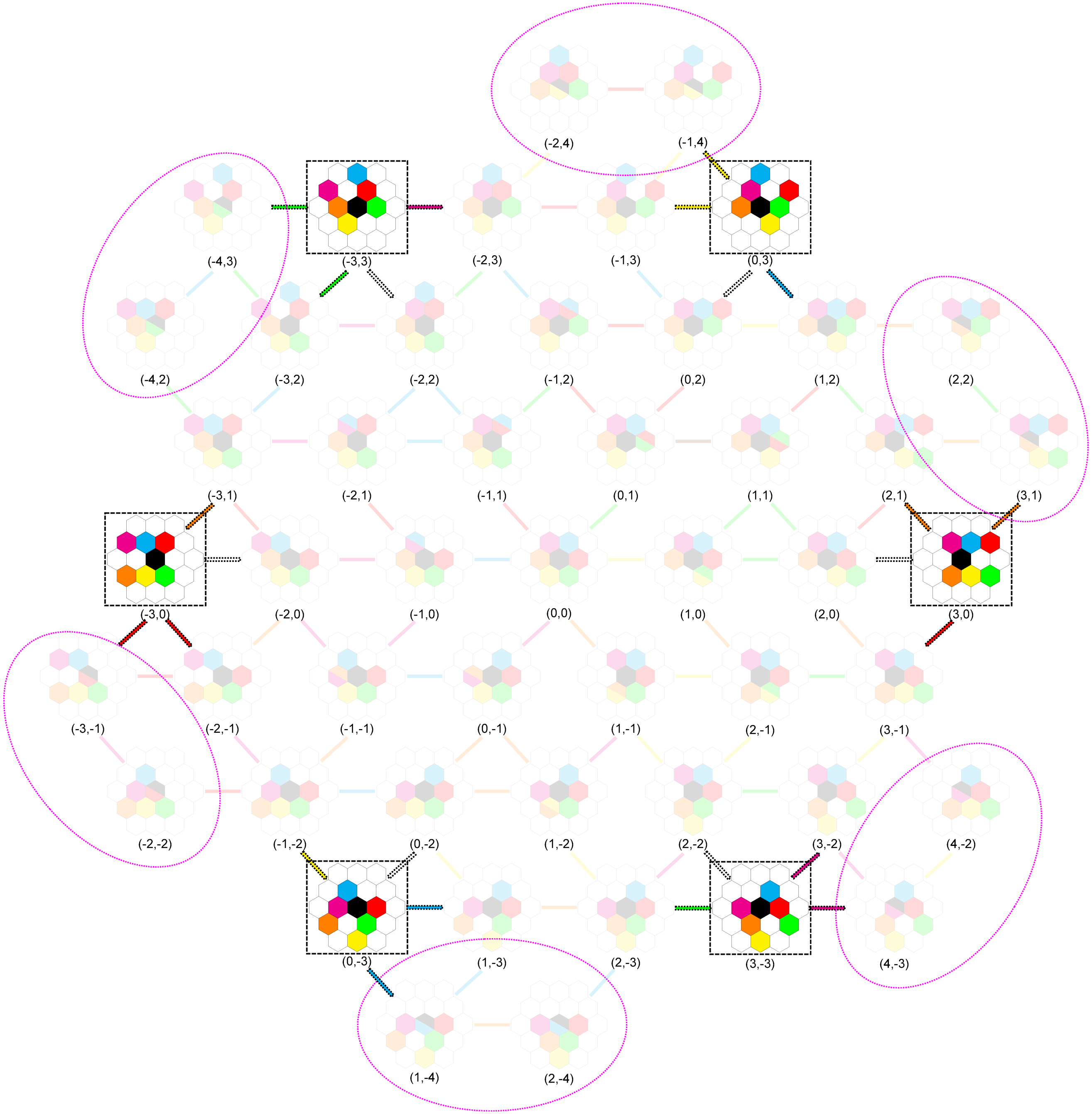
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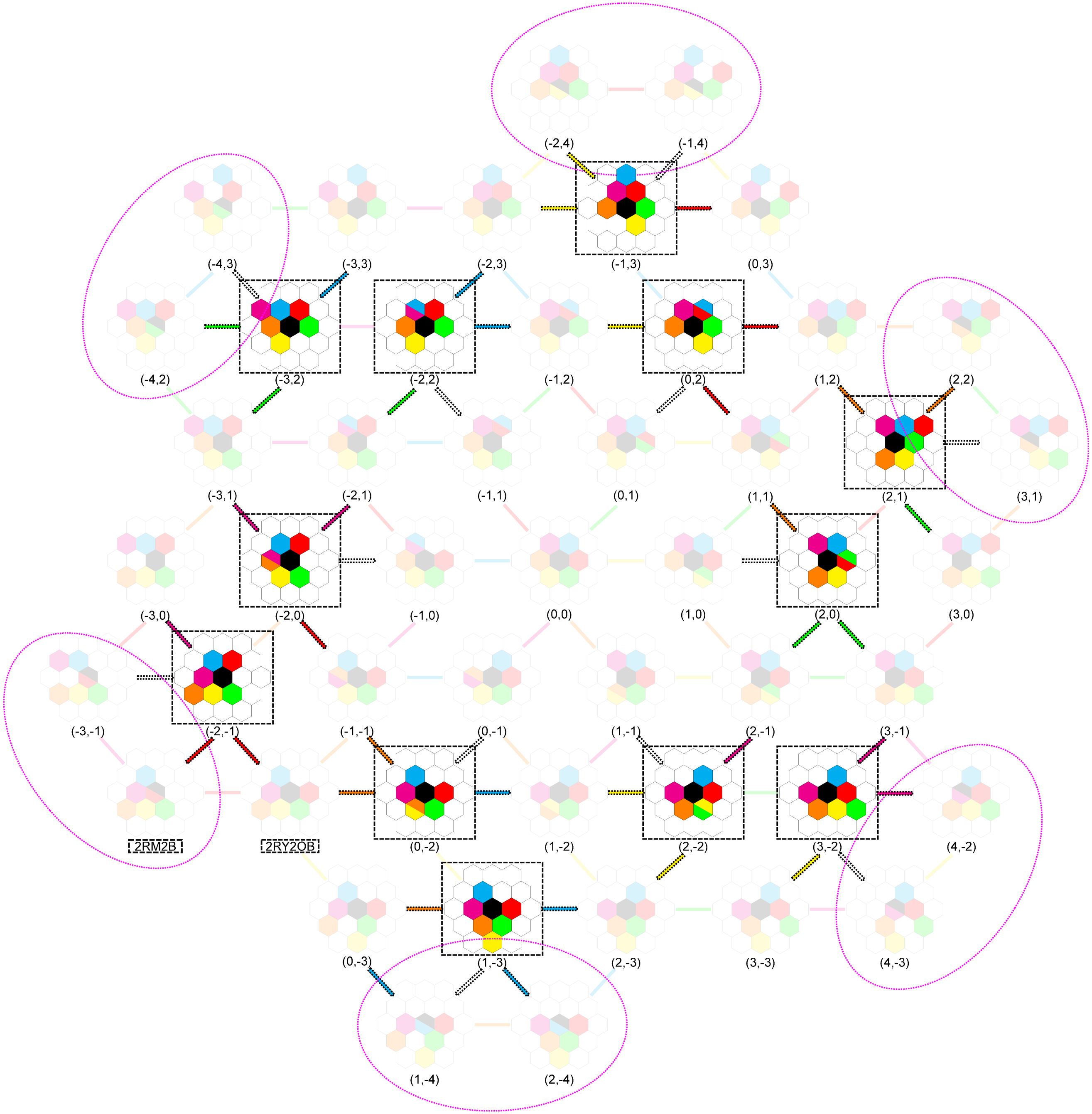

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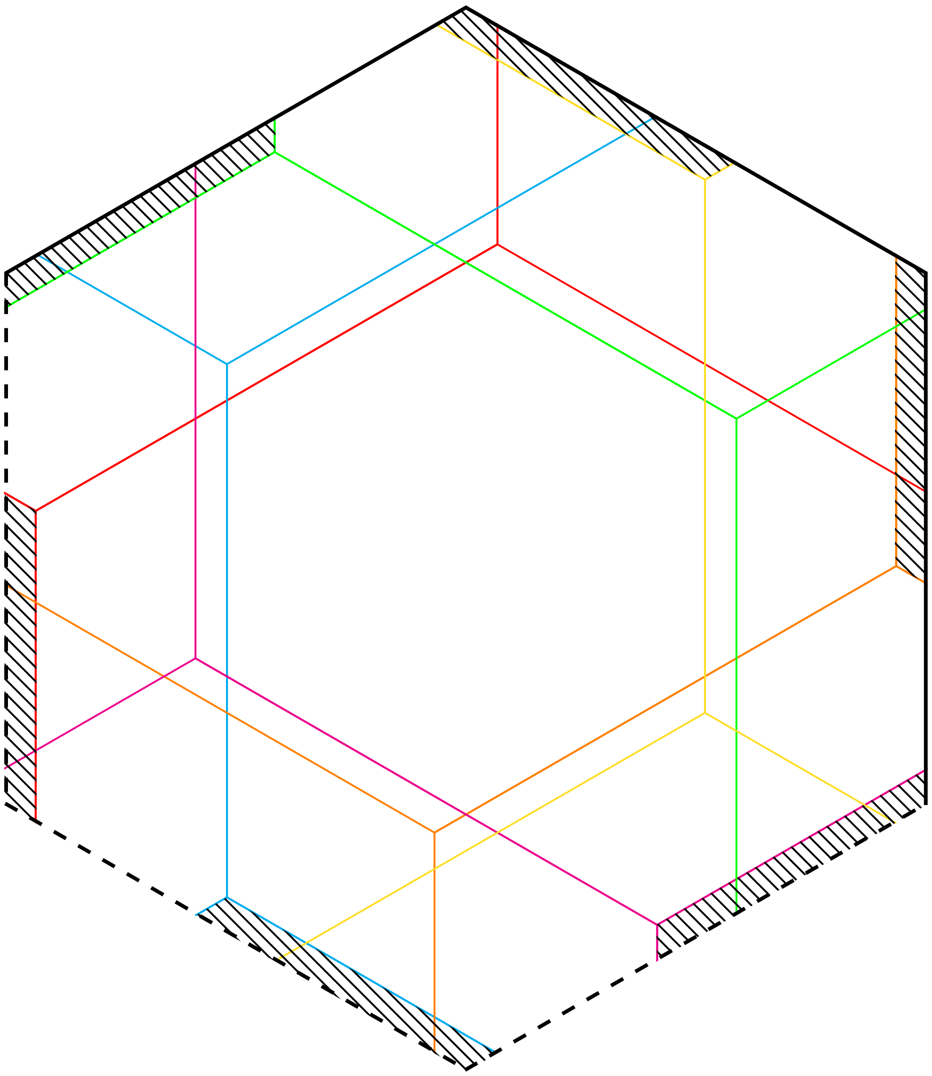
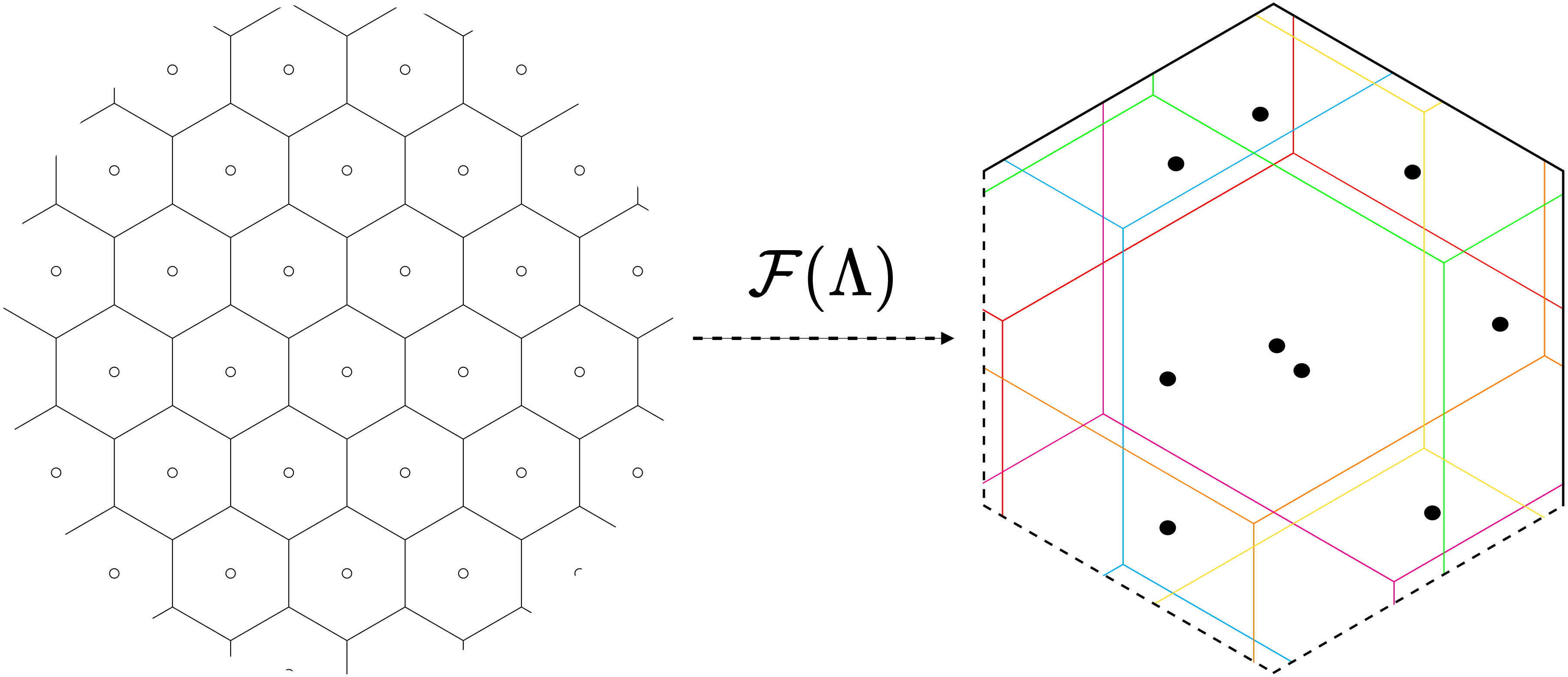
Non-injective Digitized Rigid Motions

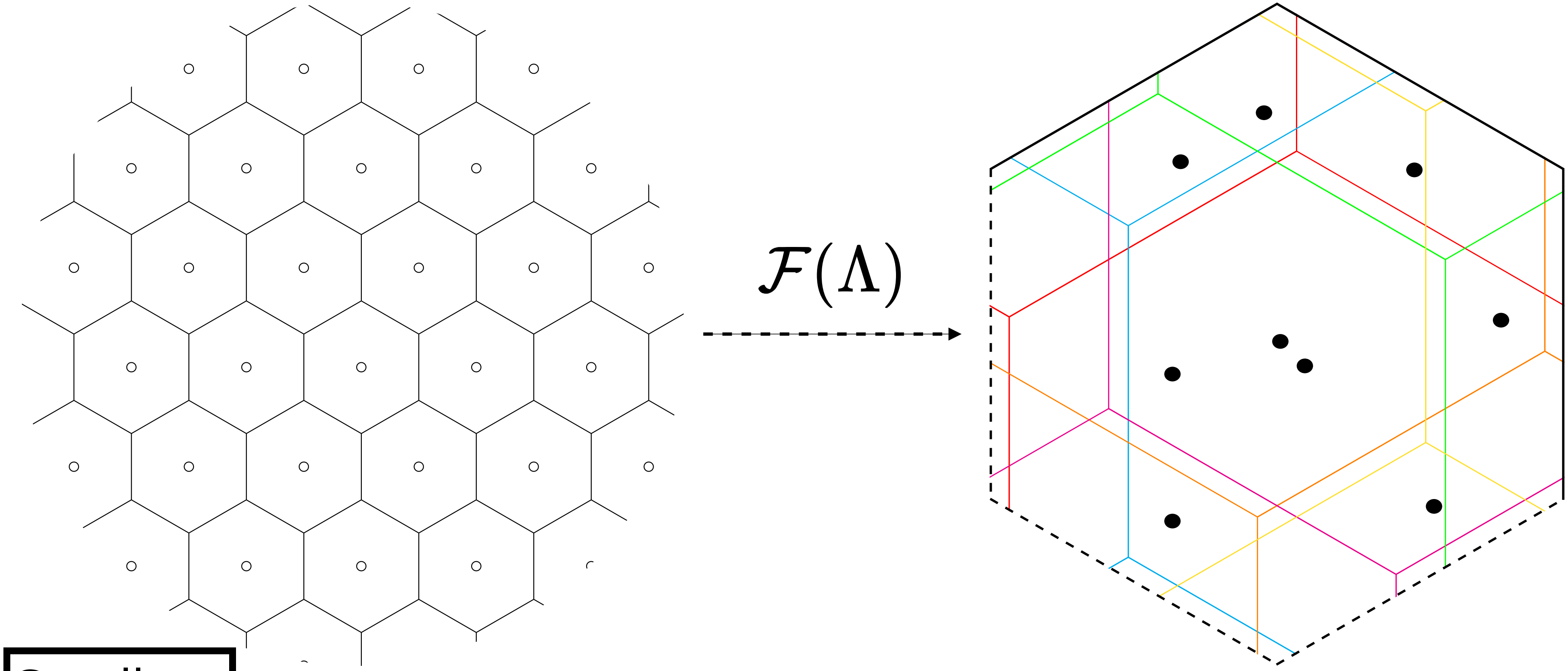


Eisenstein Rational Rotations



For what kind of parameters has the remainder map a finite number of images?

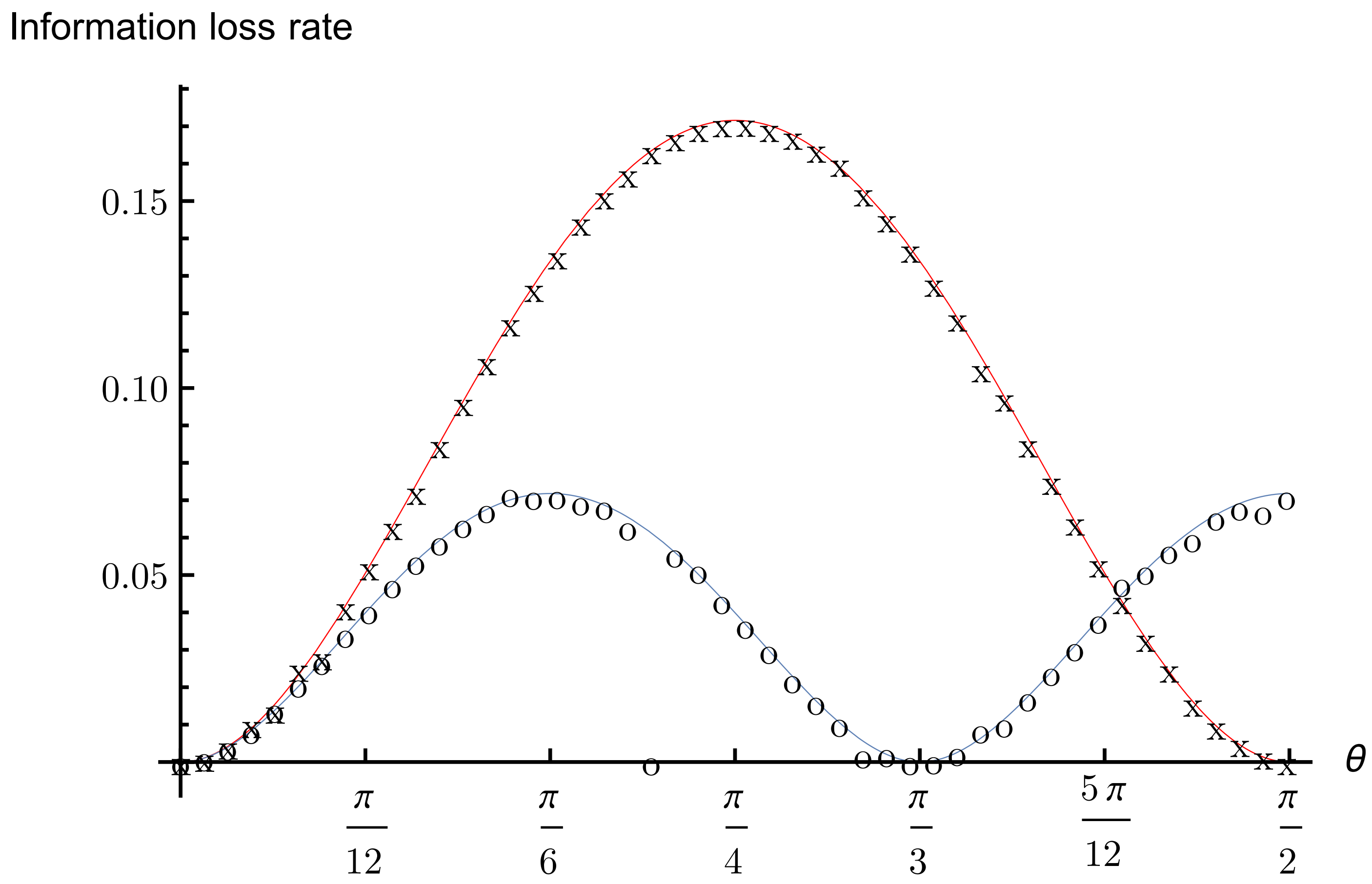
Eisenstein Rational Rotations



Corollary

If $\cos\theta = \frac{2a-b}{2c}$ and $\sin\theta = \frac{\sqrt{3}b}{2c}$ where $(a,b,c) \in \mathbb{Z}^3, \gcd(a,b,c) = 1$ and $0 < a < c < b$, then the remainder map has a finite number of images.

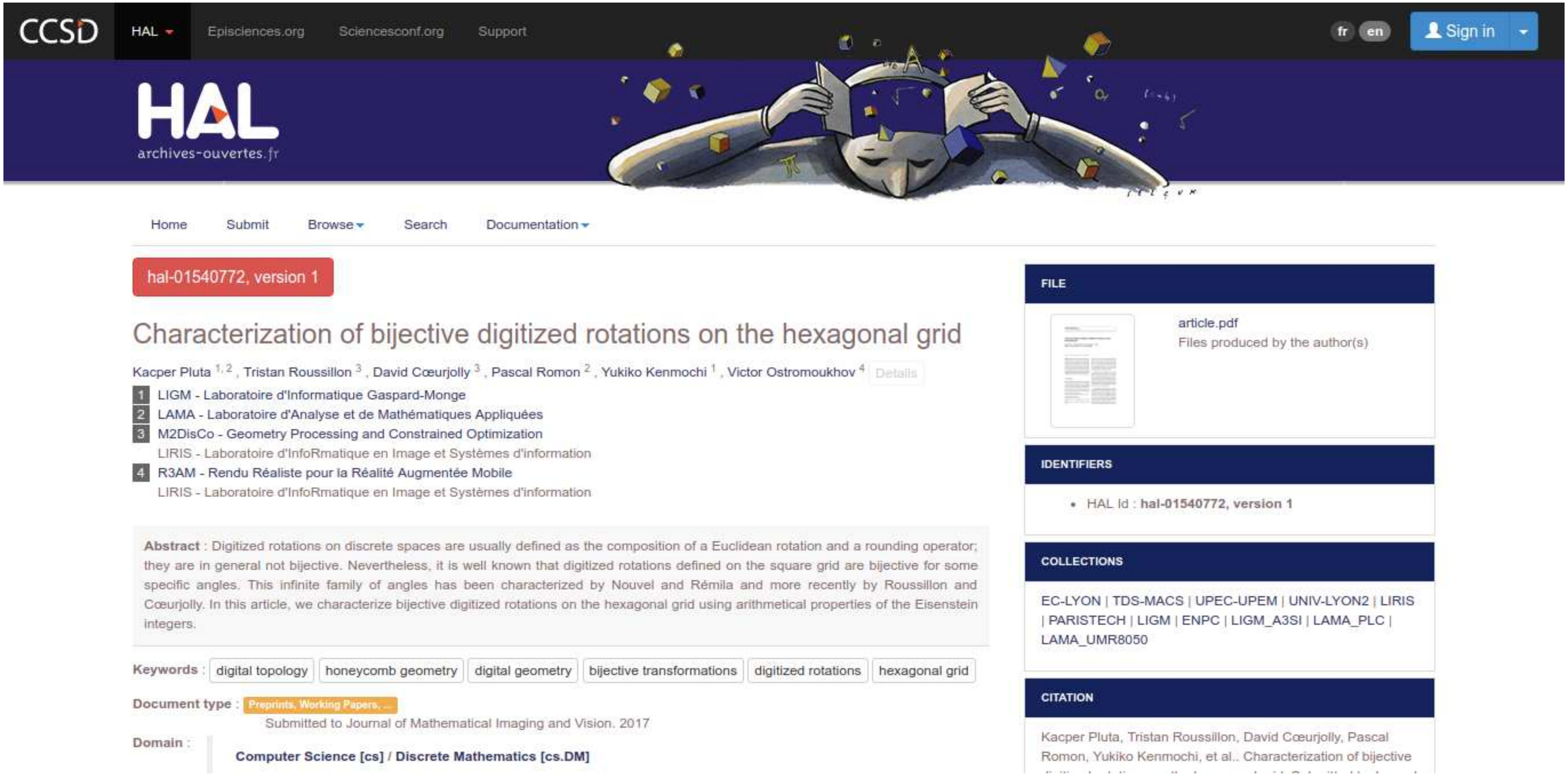
Loss of Information



Conclusions & Perspectives

- An extension of a framework to study digitized rigid motions
- Characterization of rational rotations
- We have showed that the loss of information is relatively lower for digitized rigid motions defined on the hexagonal grid
- Our tools on BSD-3 license: https://github.com/copyme/NeighborhoodMotionMapsTools

hal.archives-ouvertes.fr/hal-01540772



hal-01540772, version 1

Characterization of bijective digitized rotations on the hexagonal grid

Kacper Pluta [1,2], Tristan Roussillon [3], David Cœurjolly [3], Pascal Romon [2], Yukiko Kenmochi [1], Victor Ostromoukhov [4] Details

1 LIGM - Laboratoire d'Informatique Gaspard-Monge
2 LAMA - Laboratoire d'Analyse et de Mathématiques Appliquées
3 M2DisCo - Geometry Processing and Constrained Optimization
LIRIS - Laboratoire d'InfoRmatique en Image et Systèmes d'information
4 R3AM - Rendu Réaliste pour la Réalité Augmentée Mobile
LIRIS - Laboratoire d'InfoRmatique en Image et Systèmes d'information

Abstract : Digitized rotations on discrete spaces are usually defined as the composition of a Euclidean rotation and a rounding operator; they are in general not bijective. Nevertheless, it is well known that digitized rotations defined on the square grid are bijective for some specific angles. This infinite family of angles has been characterized by Nouvel and Rémila and more recently by Roussillon and Cœurjolly. In this article, we characterize bijective digitized rotations on the hexagonal grid using arithmetical properties of the Eisenstein integers.

Keywords : digital topology | honeycomb geometry | digital geometry | bijective transformations | digitized rotations | hexagonal grid

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Domain : Computer Science [cs] / Discrete Mathematics [cs.DM]

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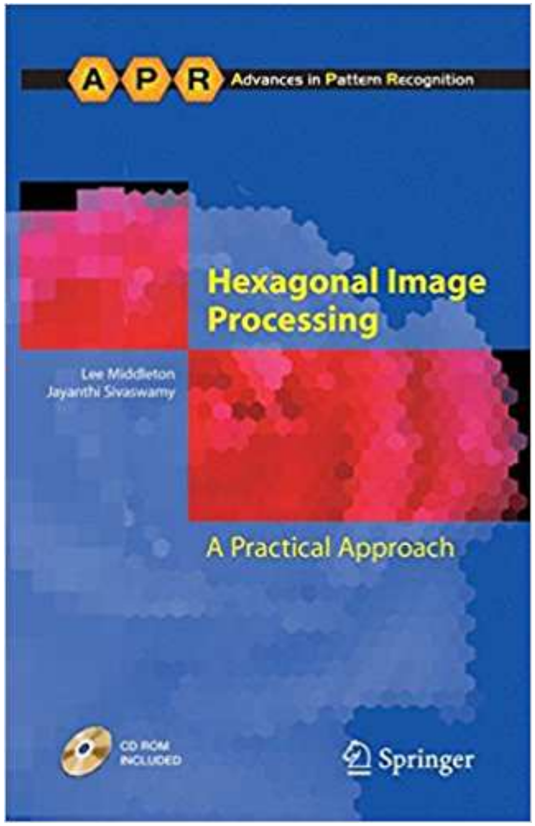
CITATION
Kacper Pluta, Tristan Roussillon, David Cœurjolly, Pascal Romon, Yukiko Kenmochi, et al.. Characterization of bijective

Homework



If you want to get into the honey business, then this book is an obligatory lecture: Middleton, Lee, and Jayanthi Sivaswamy. *Hexagonal image processing: A practical approach*. Springer Science & Business Media, 2006.